A Method Overcoming Induction During Cut-elimination

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Introduction

Aim

- Proof mining.
- Extraction of explicit information from proofs.
- Via cut-elimination: the removal of lemmas in proofs.

Cut-Elimination and Induction

- Induction: an infinitary modus ponens rule.
- Cut-elimination in the presence of induction: not possible.
- A solution: avoid induction using schemata.

Extension of **LK**

- Induction rule:

$$\frac{\Gamma \vdash \Delta, A(\bar{0}) \qquad \Pi, A(\alpha) \vdash \Lambda, A(s(\alpha))}{\Gamma, \Pi \vdash \Delta, \Lambda, A(t)}\, ind$$

- Equational rule:

$$\frac{S[t]}{S[t']}\, \mathcal{E}$$

with the condition that an equational theory $\mathcal{E} \models t = t'$.

A Motivating Example

- $\mathcal{E} = \{\hat{f}(0,x) = x,\ \hat{f}(s(n),x) = f(\hat{f}(n,x))\}$.
- $\mathcal{E} \models \hat{f}(n,x) = f^{n}(x)$.
- We prove S:

$$(\forall x)(P(x) \Rightarrow P(f(x))) \vdash$$
$$(\forall n)((P(\hat{f}(n,c)) \Rightarrow P(g(n,c))) \Rightarrow (P(c) \Rightarrow P(g(n,c))))$$

A Motivating Example (ctd.)

- φ is:

$$\dfrac{\begin{matrix}\psi\\ (\forall x)(P(x)\Rightarrow P(f(x)))\vdash C\end{matrix}\qquad\qquad\begin{matrix}(1)\\ C\vdash(\forall n)((P(\hat{f}(n,c))\Rightarrow P(g(n,c)))\Rightarrow(P(c)\Rightarrow P(g(n,c))))\end{matrix}}{(\forall x)(P(x)\Rightarrow P(f(x)))\vdash(\forall n)((P(\hat{f}(n,c))\Rightarrow P(g(n,c)))\Rightarrow(P(c)\Rightarrow P(g(n,c))))}\;cut$$

- $C=(\forall n)(\forall x)(P(x)\Rightarrow P(\hat{f}(n,x)))$ and (1) is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(c)\vdash P(c)\qquad\dfrac{P(\hat{f}(\beta,c))\vdash P(\hat{f}(\beta,c))\qquad P(g(\beta,c))\vdash P(g(\beta,c))}{P(\hat{f}(\beta,c))\Rightarrow P(g(\beta,c)),P(\hat{f}(\beta,c))\vdash P(g(\beta,c))}\;\Rightarrow:l}{P(c),P(\hat{f}(\beta,c))\Rightarrow P(g(\beta,c)),P(c)\Rightarrow P(\hat{f}(\beta,c))\vdash P(g(\beta,c))}\;\Rightarrow:l}{P(\hat{f}(\beta,c))\Rightarrow P(g(\beta,c)),P(c)\Rightarrow P(\hat{f}(\beta,c))\vdash P(c)\Rightarrow P(g(\beta,c))}\;\Rightarrow:r}{P(c)\Rightarrow P(\hat{f}(\beta,c))\vdash(P(\hat{f}(\beta,c))\Rightarrow P(g(\beta,c)))\Rightarrow(P(c)\Rightarrow P(g(\beta,c)))}\;\Rightarrow:r}{(\forall n)(\forall x)(P(x)\Rightarrow P(\hat{f}(n,x)))\vdash(P(\hat{f}(\beta,c))\Rightarrow P(g(\beta,c)))\Rightarrow(P(c)\Rightarrow P(g(\beta,c)))}\;\forall:l*}{(\forall n)(\forall x)(P(x)\Rightarrow P(\hat{f}(n,x)))\vdash(\forall n)((P(\hat{f}(n,c))\Rightarrow P(g(n,c)))\Rightarrow(P(c)\Rightarrow P(g(n,c))))}\;\forall:r$$

A Motivating Example (ctd.)

- ψ is:

$$
\begin{prooftree}
\AxiomC{$P(\hat{f}(\bar{0},u)) \vdash P(\hat{f}(\bar{0},u))$}
\RightLabel{$\mathcal{E}$}
\UnaryInfC{$P(u) \vdash P(\hat{f}(\bar{0},u))$}
\RightLabel{$\Rightarrow: r$}
\UnaryInfC{$\vdash P(u) \Rightarrow P(\hat{f}(\bar{0},u))$}
\RightLabel{$\forall: r$}
\UnaryInfC{$\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(\bar{0},x)))$}
\AxiomC{$\diagdown\widetilde{(2)}\diagup$}
\noLine
\UnaryInfC{$A, (\forall x)(P(x) \Rightarrow P(\hat{f}(\alpha,x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(s(\alpha),x)))$}
\RightLabel{ind}
\BinaryInfC{$(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(\gamma,x)))$}
\RightLabel{$\forall: r$}
\UnaryInfC{$(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall n)(\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))$}
\end{prooftree}
$$

- $A = (\forall x)(P(x) \Rightarrow P(f(x)))$ and (2) is:

$$
\begin{prooftree}
\AxiomC{$P(u) \vdash P(u)$}
\AxiomC{$P(\hat{f}(\alpha,u)) \vdash P(\hat{f}(\alpha,u))$}
\AxiomC{$P(\hat{f}(s(\alpha),u)) \vdash P(\hat{f}(s(\alpha),u))$}
\RightLabel{$\mathcal{E}$}
\UnaryInfC{$P(f(\hat{f}(\alpha,u))) \vdash P(\hat{f}(s(\alpha),u))$}
\RightLabel{$\Rightarrow: l$}
\BinaryInfC{$P(\hat{f}(\alpha,u)) \Rightarrow P(f(\hat{f}(\alpha,u))), P(\hat{f}(\alpha,u)) \vdash P(\hat{f}(s(\alpha),u))$}
\RightLabel{$\forall: l$}
\UnaryInfC{$(\forall x)(P(x) \Rightarrow P(f(x))), P(\hat{f}(\alpha,u)) \vdash P(\hat{f}(s(\alpha),u))$}
\RightLabel{$\Rightarrow: l$}
\BinaryInfC{$P(u), (\forall x)(P(x) \Rightarrow P(f(x))), P(u) \Rightarrow P(\hat{f}(\alpha,u)) \vdash P(\hat{f}(s(\alpha),u))$}
\RightLabel{$\Rightarrow: r$}
\UnaryInfC{$(\forall x)(P(x) \Rightarrow P(f(x))), P(u) \Rightarrow P(\hat{f}(\alpha,u)) \vdash P(u) \Rightarrow P(\hat{f}(s(\alpha),u))$}
\RightLabel{$\forall: l$}
\UnaryInfC{$(\forall x)(P(x) \Rightarrow P(f(x))), (\forall x)(P(x) \Rightarrow P(\hat{f}(\alpha,x))) \vdash P(u) \Rightarrow P(\hat{f}(s(\alpha),u)))$}
\RightLabel{$\forall: r$}
\UnaryInfC{$(\forall x)(P(x) \Rightarrow P(f(x))), (\forall x)(P(x) \Rightarrow P(\hat{f}(\alpha,x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(s(\alpha),x)))$}
\end{prooftree}
$$

A Motivating Example (ctd.)

- After some reduction steps:

$$\cfrac{\cfrac{\cfrac{\widetilde{\diagdown\,\psi\,\diagup}}{A \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(\beta,x)))}\,\mathit{ind} \qquad \begin{array}{c}\widetilde{\diagdown\,(1')\,\diagup}\\ (\forall x)(P(x) \Rightarrow P(\hat{f}(\beta,x))) \vdash B\end{array}}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (P(\hat{f}(\beta,c)) \Rightarrow P(g(\beta,c))) \Rightarrow (P(c) \Rightarrow P(g(\beta,c)))}\,\mathit{cut}}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall n)((P(\hat{f}(n,c)) \Rightarrow P(g(n,c))) \Rightarrow (P(c) \Rightarrow P(g(n,c))))}\,\forall : r$$

- Cannot proceed!

- In fact, there is no cut-free proof of S, induction on $(\forall n)((P(\hat{f}(n,c)) \Rightarrow P(g(n,c))) \Rightarrow (P(c) \Rightarrow P(g(n,c))))$ fails.

A Motivating Example (ctd.)

- The sequents S_n:

$$(\forall x)(P(x) \Rightarrow P(f(x))) \vdash$$
$$(P(\hat{f}(\bar{n}, c)) \Rightarrow P(g(\bar{n}, c))) \Rightarrow (P(c) \Rightarrow P(g(\bar{n}, c)))$$

do have cut-free proofs for all $\bar{n}$.

- Uniform description of the sequence of cut-free proofs is needed.
- Develop machinery to obtain such a description.

Schematic Proof Systems

Language

- Consider two sorts ω, ι.
- Our language consists of:
  - arithmetical variables $i, j, k, n \colon \omega$,
  - first-order variables $x, y, z \colon \iota$,
  - schematic variables $u, v \colon \omega \to \iota$,
  - constant function symbols $f, g \colon \tau_1 \times \cdots \times \tau_n \to \tau$,
  - defined function symbols $\hat{f}, \hat{g} \colon \omega \times \tau_1 \times \cdots \times \tau_n \to \tau$,
  - predicate symbols P, Q and the logical connectives $\neg, \wedge, \vee, \Rightarrow, \forall, \exists, \bigwedge, \bigvee$.

Language (ctd.)

- Terms are defined in usual inductive fashion using variables and constant function symbols.
- Arithmetical terms are subset of terms constructed using $0\colon \omega, s\colon \omega \to \omega, +\colon \omega \times \omega \to \omega$ and arithmetical variables.
- Formulas are defined in usual inductive fashion using predicate symbols and connectives $\neg, \wedge, \vee, \Rightarrow, \forall, \exists$ (quantification is allowed only on first-order variables).

Language (ctd.)

- Term schemata: terms and primitive recursion on terms using defined function symbols, i.e. for every $\hat{f}$:

$$\begin{array}{rcl} \hat{f}(0, x_1, \ldots, x_n) & \rightarrow & s, \\ \hat{f}(k+1, x_1, \ldots, x_n) & \rightarrow & t[\hat{f}(k, x_1, \ldots, x_n)] \end{array}$$

s.t. $V(s) \cup V(t) = \{x_1, \ldots, x_n\}$ and s, t are terms.

- Example: $\hat{f}(n, x)$ defining $f^n(x)$:

$$\begin{array}{rcl} \hat{f}(0, x) & \rightarrow & x, \\ \hat{f}(k+1, x) & \rightarrow & f(\hat{f}(k, x)). \end{array}$$

Language (ctd.)

- Formula schemata: formulas are formula schemata and if A is a formula schema, then $\bigwedge_{i=a}^{b} A$ and $\bigvee_{i=a}^{b} A$ are formula schemata as well.
- Example: $(\exists y)(\bigvee_{i=0}^{n}(\forall x)A(i,x,y))$ defining $(\exists y)((\forall x)A(0,x,y) \vee \cdots \vee (\forall x)A(n,x,y))$ which is equivalent to $(\exists y)((\forall x_0)A(0,x_0,y) \vee \cdots \vee (\forall x_n)A(n,x_n,y))$.

Calculus $\mathbf{LK}_s$

- Sequent: expression $S(x_1, \ldots, x_\alpha) : \Gamma \vdash \Delta$.
- Proof link: expression $\dfrac{(\varphi(a_1, \ldots, a_\alpha))}{S(a_1, \ldots, a_\alpha)}$
- Axioms: proof links or $A \vdash A$.
- Usual **LK** rules operating on formula schemata and the $\mathcal{E}$ rule.

Proof Schema

- Tuple of pairs of $\mathbf{LK}_s$-proofs.
- Each pair is associated with one proof symbol.
- Each pair corresponds to a base and step case of inductive definition.
- The proof symbols are ordered.

An Example

- Let $\hat{f} : \omega \times \iota \to \iota$ and define $\hat{f}(n, x)$ as:

$$\begin{array}{rcl} \hat{f}(0, x) & \to & x, \\ \hat{f}(k+1, x) & \to & f(\hat{f}(k, x)) \end{array}$$

- Let $\Psi = \langle \phi, \psi \rangle$ be a proof schema of

$$(\forall x)(P(x) \Rightarrow P(f(x))) \vdash$$
$$(P(\hat{f}(n, c)) \Rightarrow P(g(n, c))) \Rightarrow (P(c) \Rightarrow P(g(n, c)))$$

An Example (ctd.)

- φ is associated with the pair: step case

$$\dfrac{\dfrac{(\psi(k+1))}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x)))} \qquad \overset{\frown}{\backslash (1) /}}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\ cut$$

where (1) is:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(c) \vdash P(c) \qquad \dfrac{P(\hat{f}(k+1,c)) \vdash P(\hat{f}(k+1,c)) \qquad P(g(k+1,c)) \vdash P(g(k+1,c))}{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))}\ \Rightarrow: l}{P(c), P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))}\ \Rightarrow: l}{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(c) \Rightarrow P(g(k+1,c))}\ \Rightarrow: r}{P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\ \Rightarrow: r}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\ \forall: l$$

An Example (ctd.)

- ψ is associated with the pair: base case (induction basis)

$$\dfrac{\dfrac{\dfrac{\dfrac{P(\hat{f}(0, u(0))) \vdash P(\hat{f}(0, u(0)))}{P(u(0)) \vdash P(\hat{f}(0, u(0)))}\,\mathcal{E}}{\vdash P(u(0)) \Rightarrow P(\hat{f}(0, u(0)))}\,\Rightarrow : r}{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(0, x)))}\,\forall : r}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(0, x)))}\,w : l$$

- and step case

$$\dfrac{\dfrac{(\psi(k))}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k, x)))} \qquad \diagdown\widetilde{(2)}\diagup}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k+1, x)))}\,cut, c : l$$

An Example (ctd.)

where (2) is (induction step):

$$
\begin{prooftree}
\AxiomC{$P(u(sk)) \vdash P(u(sk))$}
\AxiomC{$P(\hat{f}(k, u(sk))) \vdash P(\hat{f}(k, u(sk)))$}
\AxiomC{$P(\hat{f}(sk, u(sk))) \vdash P(\hat{f}(sk, u(sk)))$}
\RightLabel{$\mathcal{E}$}
\UnaryInfC{$P(f(\hat{f}(k, u(sk)))) \vdash P(\hat{f}(sk, u(sk)))$}
\RightLabel{$\Rightarrow: l$}
\BinaryInfC{$P(\hat{f}(k, u(sk))), P(\hat{f}(k, u(sk))) \Rightarrow P(f(\hat{f}(k, u(sk)))) \vdash P(\hat{f}(sk, u(sk)))$}
\RightLabel{$\forall: l$}
\UnaryInfC{$P(\hat{f}(k, u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk, u(sk)))$}
\RightLabel{$\Rightarrow: l$}
\BinaryInfC{$P(u(sk)), P(u(sk)) \Rightarrow P(\hat{f}(k, u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk, u(sk)))$}
\RightLabel{$\Rightarrow: r$}
\UnaryInfC{$P(u(sk)) \Rightarrow P(\hat{f}(k, u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk, u(sk)))$}
\RightLabel{$\forall: l$}
\UnaryInfC{$(\forall x)(P(x) \Rightarrow P(\hat{f}(k, x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk, u(sk)))$}
\RightLabel{$\forall: r$}
\UnaryInfC{$(\forall x)(P(x) \Rightarrow P(\hat{f}(k, x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(sk, x)))$}
\end{prooftree}
$$

Key Results

Proposition (Soundness)

$\mathbf{LK}_s$ *is sound.*

Proposition (Correspondence)

*Let ϕ be an **LK**-proof without nested inductions of a sequent S fulfilling the following conditions:*

- *The inductions occurring in ϕ are standard inductions on the natural numbers.*
- *The term definitions in S are (primitive) recursive.*

Then ϕ can be transformed into a proof schema.

Resolution Calculus $\mathcal{R}_s$

- Clauses and clause schemata.
- Clause-set terms and clause set schemata.
- Resolution terms and resolution proof schemata.
- Substitution schema.

Key Results

Proposition (Soundness)

$\mathcal{R}_s$ is sound.

Proposition

Unification problem is undecidable for term schemata.

Cut-Elimination in Proof Schemata

Cut-Elimination Methods

- Reductive methods: fail, since cannot shift cuts over proof links.
- Cut-Elimination by RESolution: analyses all cuts together, no shifts needed.

Key Points of CERES

- A clause set, called characteristic clause set.
- A set of cut-free proofs, called (set of) projections.
- A Refutation of the characteristic clause set.

Key Results About CERES

Proposition (Baaz & Leitsch, 2001)

*Let π be an **LK**-proof. Then $\mathrm{CL}(\pi)$ is unsatisfiable.*

Proposition (Baaz & Leitsch, 2001)

*Let π be an **LK**-proof with end-sequent S, then for all clauses $C \in \mathrm{CL}(\pi)$, there exists an **LK**-proof $\pi_C \in PR(\pi)$ with end-sequent $S \circ C$.*

The *CERES*$_s$ Method

- Configuration Ω of ψ is a set of formula occurrences from the end-sequent of ψ.
- $\mathrm{cl}^{\psi,\Omega}$ is a unique symbol, called *clause-set symbol*.
- $\mathrm{pr}^{\psi,\Omega}$ is a unique symbol, called *projection symbol*.

The *CERES*$_s$ Method (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.
- Base case for ψ:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(\hat{f}(0,u(0))) \vdash P(\hat{f}(0,u(0)))}{P(u(0)) \vdash P(\hat{f}(0,u(0)))}\,\mathcal{E}}{\vdash P(u(0)) \Rightarrow P(\hat{f}(0,u(0)))}\,\Rightarrow: r}{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(0,x)))}\,\forall: r}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(0,x)))}\,w: l$$

The $CERES_s$ Method (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.

- Base case for ψ:

$$\cfrac{\cfrac{\cfrac{\cfrac{P(\hat{f}(0,u(0))) \vdash P(\hat{f}(0,u(0)))}{P(u(0)) \vdash P(\hat{f}(0,u(0)))}\,\mathcal{E}}{\vdash P(u(0)) \Rightarrow P(\hat{f}(0,u(0)))}\,\Rightarrow: r}{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(0,x)))}\,\forall: r}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(0,x)))}\,w: l$$

$\mathrm{cl}^{\psi,\Omega}(0) = \{P(\hat{f}(0,u(0))) \vdash P(\hat{f}(0,u(0)))\}$ and

$\mathrm{pr}^{\psi,\Omega}(0) = w_l(P(\hat{f}(0,u(0))) \vdash P(\hat{f}(0,u(0))))$

The *CERES*$_s$ Method (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.

- Step case for ψ:

$$\dfrac{\dfrac{(\psi(k))}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k,x)))} \qquad \widetilde{\backslash (2) /}}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x)))}\ cut, c: l$$

The $CERES_s$ Method (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.
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$$\mathrm{cl}^{\psi,\Omega}(k+1) = \mathrm{cl}^{\psi,\Omega}(k) \oplus$$

The $CERES_s$ Method (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.

- (2) is:

$$\dfrac{\dfrac{P(\hat{f}(k,u(sk))) \vdash P(\hat{f}(k,u(sk))) \qquad \dfrac{P(\hat{f}(sk,u(sk))) \vdash P(\hat{f}(sk,u(sk)))}{P(f(\hat{f}(k,u(sk)))) \vdash P(\hat{f}(sk,u(sk)))}\,\mathcal{E}}{\dfrac{P(\hat{f}(k,u(sk))), P(\hat{f}(k,u(sk))) \Rightarrow P(f(\hat{f}(k,u(sk)))) \vdash P(\hat{f}(sk,u(sk)))}{P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk)))}\,\forall: l}\,\Rightarrow: l$$

$$\dfrac{\dfrac{\dfrac{\dfrac{P(u(sk)) \vdash P(u(sk)) \qquad P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk)))}{P(u(sk)), P(u(sk)) \Rightarrow P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk)))}\,\Rightarrow: l}{P(u(sk)) \Rightarrow P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk,u(sk)))}\,\Rightarrow: r}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k,x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk,u(sk)))}\,\forall: l}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k,x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(sk,x)))}\,\forall: r$$

$$\mathrm{cl}^{\psi,\Omega}(k+1) = \mathrm{cl}^{\psi,\Omega}(k) \oplus$$

The $CERES_s$ Method (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.
- (2) is:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(u(sk)) \vdash P(u(sk)) \qquad \dfrac{\dfrac{P(\hat{f}(k,u(sk))) \vdash P(\hat{f}(k,u(sk))) \qquad \dfrac{P(\hat{f}(sk,u(sk))) \vdash P(\hat{f}(sk,u(sk)))}{P(f(\hat{f}(k,u(sk)))) \vdash P(\hat{f}(sk,u(sk)))}\;\mathcal{E}}{P(\hat{f}(k,u(sk))), P(\hat{f}(k,u(sk))) \Rightarrow P(f(\hat{f}(k,u(sk)))) \vdash P(\hat{f}(sk,u(sk)))}\;\Rightarrow: l}{P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk)))}\;\forall: l}{P(u(sk)), P(u(sk)) \Rightarrow P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk)))}\;\Rightarrow: l}{P(u(sk)) \Rightarrow P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk,u(sk)))}\;\Rightarrow: r}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k,x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk,u(sk)))}\;\forall: l}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k,x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(sk,x)))}\;\forall: r$$

$$\begin{aligned}\mathrm{cl}^{\psi,\Omega}(k+1) = \mathrm{cl}^{\psi,\Omega}(k) \oplus \{P(u(sk)) \vdash P(u(sk))\} \oplus \\ (\{P(\hat{f}(k,u(sk))) \vdash\} \otimes \{\vdash P(\hat{f}(sk,u(sk)))\})\end{aligned}$$

The *CERES*$_s$ Method (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.

- Step case for ψ:

$$\cfrac{\overset{(\psi(k))}{\overline{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k,x)))}} \qquad \widetilde{\backslash (2) /}}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x)))}\ cut, c: l$$

The *CERES*$_s$ Method (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.
- Step case for ψ:

$$\dfrac{\dfrac{(\psi(k))}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k,x)))} \qquad \widetilde{\backslash (2) /}}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x)))}\ cut, c: l$$

$$\mathrm{pr}^{\psi,\Omega}(k+1) = c_l(w^{A\vdash}(\mathrm{pr}^{\psi,\Omega}(k)) \oplus w^{A\vdash}(\qquad\qquad))$$

The *CERES$_s$* Method (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.
- (2) is:

$$
\begin{prooftree}
\AxiomC{$P(u(sk)) \vdash P(u(sk))$}
\AxiomC{$P(\hat{f}(k,u(sk))) \vdash P(\hat{f}(k,u(sk)))$}
\AxiomC{$P(\hat{f}(sk,u(sk))) \vdash P(\hat{f}(sk,u(sk)))$}
\RightLabel{$\mathcal{E}$}
\UnaryInfC{$P(f(\hat{f}(k,u(sk)))) \vdash P(\hat{f}(sk,u(sk)))$}
\RightLabel{$\Rightarrow: l$}
\BinaryInfC{$P(\hat{f}(k,u(sk))), P(\hat{f}(k,u(sk))) \Rightarrow P(f(\hat{f}(k,u(sk)))) \vdash P(\hat{f}(sk,u(sk)))$}
\RightLabel{$\forall: l$}
\UnaryInfC{$P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk)))$}
\RightLabel{$\Rightarrow: l$}
\BinaryInfC{$P(u(sk)), P(u(sk)) \Rightarrow P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk)))$}
\RightLabel{$\Rightarrow: r$}
\UnaryInfC{$P(u(sk)) \Rightarrow P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk,u(sk)))$}
\RightLabel{$\forall: l$}
\UnaryInfC{$(\forall x)(P(x) \Rightarrow P(\hat{f}(k,x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk,u(sk)))$}
\RightLabel{$\forall: r$}
\UnaryInfC{$(\forall x)(P(x) \Rightarrow P(\hat{f}(k,x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(sk,x)))$}
\end{prooftree}
$$

$$\mathrm{pr}^{\psi,\Omega}(k+1) = c_l(w^{A\vdash}(\mathrm{pr}^{\psi,\Omega}(k)) \oplus w^{A\vdash}(\qquad))$$

The $CERES_s$ Method (ctd.)

- (2) is:

$$\dfrac{\dfrac{P(u(sk)) \vdash P(u(sk)) \qquad \dfrac{\dfrac{P(\hat{f}(k,u(sk))) \vdash P(\hat{f}(k,u(sk))) \qquad \dfrac{P(\hat{f}(sk,u(sk))) \vdash P(\hat{f}(sk,u(sk)))}{P(f(\hat{f}(k,u(sk)))) \vdash P(\hat{f}(sk,u(sk)))}\,\mathcal{E}}{P(\hat{f}(k,u(sk))), P(\hat{f}(k,u(sk))) \Rightarrow P(f(\hat{f}(k,u(sk)))) \vdash P(\hat{f}(sk,u(sk)))}\Rightarrow: l}{P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk)))}\,\forall: l}{\dfrac{\dfrac{P(u(sk)), P(u(sk)) \Rightarrow P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk)))}{P(u(sk)) \Rightarrow P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk,u(sk)))}\Rightarrow: r}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k,x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk,u(sk)))}\,\forall: l}\Rightarrow: l}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k,x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(sk,x)))}\,\forall: r$$

$$\begin{aligned}\mathrm{pr}^{\psi,\Omega}(k+1) = c_l(w^{A\vdash}(\mathrm{pr}^{\psi,\Omega}(k)) \oplus w^{A\vdash}(\\ w^{A\vdash}(P(u(sk)) \vdash P(u(sk)))\oplus \\ w^{\vdash}(\forall_l(P(\hat{f}(k,u(sk))) \vdash P(\hat{f}(k,u(sk)))\otimes_{\Rightarrow_l} \\ P(\hat{f}(sk,u(sk))) \vdash P(\hat{f}(sk,u(sk)))))))\end{aligned}$$

The *CERES$_s$* Method (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.
- Step case for φ:

$$\dfrac{\dfrac{(\psi(k+1))}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x)))} \qquad \widetilde{\backslash (1) /}}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\ \mathit{cut}$$

The *CERES$_s$* Method (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.

- Step case for φ:

$$\cfrac{\cfrac{(\psi(k+1))}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x)))} \qquad \diagdown\widetilde{(1)}\diagup}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\ cut$$

$$\mathrm{cl}^{\varphi,\emptyset}(k+1) = \mathrm{cl}^{\psi,\Omega}(k+1) \oplus$$

The *CERES*$_s$ Method (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.
- (1) is:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(c) \vdash P(c) \qquad \dfrac{P(\hat{f}(k+1,c)) \vdash P(\hat{f}(k+1,c)) \qquad P(g(k+1,c)) \vdash P(g(k+1,c))}{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))}\;{\Rightarrow : l}}{P(c), P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))}\;{\Rightarrow : l}}{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(c) \Rightarrow P(g(k+1,c))}\;{\Rightarrow : r}}{P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\;{\Rightarrow : r}}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\;{\forall : l}$$

$$\mathrm{cl}^{\varphi,\emptyset}(k+1) = \mathrm{cl}^{\psi,\Omega}(k+1) \oplus$$

The $CERES_s$ Method (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.
- (1) is:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(c) \vdash P(c) \qquad \dfrac{P(\hat{f}(k+1,c)) \vdash P(\hat{f}(k+1,c)) \qquad P(g(k+1,c)) \vdash P(g(k+1,c))}{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))}\Rightarrow: l}{P(c), P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))}\Rightarrow: l}{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(c) \Rightarrow P(g(k+1,c))}\Rightarrow: r}{P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\Rightarrow: r}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\forall: l$$

$$\begin{aligned}\mathrm{cl}^{\varphi,\emptyset}(k+1) = \mathrm{cl}^{\psi,\Omega}(k+1) \oplus \{\vdash P(c)\} \oplus \\ (\{P(\hat{f}(k+1,c) \vdash\} \otimes \{\vdash\})\end{aligned}$$

The $CERES_s$ Method (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.
- Base case for φ:

$$\mathrm{cl}^{\varphi,\emptyset}(0) = \{P(\hat{f}(0,u(0))) \vdash P(\hat{f}(0,u(0)))\} \oplus \\ \{\vdash P(c)\} \oplus (\{P(\hat{f}(0,c) \vdash\} \otimes \{\vdash\})$$

The *CERES*$_s$ Method (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.
- Step case for φ:

$$\dfrac{\dfrac{(\psi(k+1))}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x)))} \qquad \widetilde{\diagdown\,(1)\,\diagup}}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\ \mathit{cut}$$

The $CERES_s$ Method (ctd.)

- Let $A = (\forall x)(P(x) \Rightarrow P(f(x)))$ and $B(k+1) = (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))$
- Step case for φ:

$$\dfrac{\dfrac{(\psi(k+1))}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash {\color{blue}(\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x)))}} \qquad \diagdown\widetilde{(1)}\diagup}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\ cut$$

$$\mathrm{pr}^{\varphi,\emptyset}(k+1) = w^{\vdash B(k+1)}(\mathrm{pr}^{\psi,\Omega}(k+1)) \oplus w^{A\vdash}(\qquad\qquad)$$

The $CERES_s$ Method (ctd.)

- Let $A = (\forall x)(P(x) \Rightarrow P(f(x)))$ and $B(k+1) = (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))$
- (1) is:

$$\begin{prooftree}
\AxiomC{$P(c) \vdash P(c)$}
\AxiomC{$P(\hat{f}(k+1,c)) \vdash P(\hat{f}(k+1,c))$}
\AxiomC{$P(g(k+1,c)) \vdash P(g(k+1,c))$}
\RightLabel{$\Rightarrow: l$}
\BinaryInfC{$P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))$}
\RightLabel{$\Rightarrow: l$}
\BinaryInfC{$P(c), P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))$}
\RightLabel{$\Rightarrow: r$}
\UnaryInfC{$P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(c) \Rightarrow P(g(k+1,c))$}
\RightLabel{$\Rightarrow: r$}
\UnaryInfC{$P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))$}
\RightLabel{$\forall: l$}
\UnaryInfC{$(\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))$}
\end{prooftree}$$

$$\mathrm{pr}^{\varphi,\emptyset}(k+1) = w^{\vdash B(k+1)}(\mathrm{pr}^{\psi,\Omega}(k+1)) \oplus w^{A\vdash}(\qquad\qquad)$$

The $CERES_s$ Method (ctd.)

- Let $A = (\forall x)(P(x) \Rightarrow P(f(x)))$ and $B(k+1) = (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))$
- (1) is:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(c) \vdash P(c) \qquad \dfrac{P(\hat{f}(k+1,c)) \vdash P(\hat{f}(k+1,c)) \qquad P(g(k+1,c)) \vdash P(g(k+1,c))}{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))}\,{\Rightarrow: l}}{P(c), P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))}\,{\Rightarrow: l}}{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(c) \Rightarrow P(g(k+1,c))}\,{\Rightarrow: r}}{P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\,{\Rightarrow: r}}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\,{\forall: l}$$

$$\begin{aligned}\mathrm{pr}^{\varphi,\emptyset}(k+1) = w^{\vdash B(k+1)}(\mathrm{pr}^{\psi,\Omega}(k+1)) \oplus w^{A\vdash}(\\ \Rightarrow_r (\Rightarrow_r (w^{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)) \vdash P(g(k+1,c))}(P(c) \vdash P(c)) \oplus \\ w^{P(c)\vdash}(P(\hat{f}(k+1,c)) \vdash P(\hat{f}(k+1,c)) \otimes_{\Rightarrow_l} \\ P(g(k+1,c)) \vdash P(g(k+1,c))))))\end{aligned}$$

The $CERES_s$ Method (ctd.)

- Let $A = (\forall x)(P(x) \Rightarrow P(f(x)))$
- Let $B(0) = (P(\hat{f}(0,c)) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))$
- Base case for φ:

$$
\begin{gathered}
\mathrm{pr}^{\varphi,\emptyset}(0) = w^{\vdash B(0)}(w_l(P(\hat{f}(0,u(0))) \vdash P(\hat{f}(0,u(0)))))\oplus \\
w^{A\vdash}(\Rightarrow_r (\Rightarrow_r (w^{P(\hat{f}(0,c))\Rightarrow P(g(0,c))\vdash P(g(0,c))}(P(c) \vdash P(c))\oplus \\
w^{P(c)\vdash}(P(\hat{f}(0,c)) \vdash P(\hat{f}(0,c)) \otimes_{\Rightarrow_l} P(g(0,c)) \vdash P(g(0,c))))))
\end{gathered}
$$

Characteristic Clause Set Schema

- $\mathrm{CL}(\Psi) = (\mathrm{cl}^{\varphi,\emptyset}, \mathrm{cl}^{\psi,\Omega})$ where

$$\begin{aligned}
\mathrm{cl}^{\varphi,\emptyset}(0) &\rightarrow \{P(\hat{f}(0,u(0))) \vdash P(\hat{f}(0,u(0)))\} \oplus \\
&\qquad \{\vdash P(c)\} \oplus (\{P(\hat{f}(0,c) \vdash\} \otimes \{\vdash\}) \\
\mathrm{cl}^{\varphi,\emptyset}(k+1) &\rightarrow \mathrm{cl}^{\psi,\Omega}(k+1) \oplus \{\vdash P(c)\} \oplus \\
&\qquad (\{P(\hat{f}(k+1,c) \vdash\} \otimes \{\vdash\}) \\
\mathrm{cl}^{\psi,\Omega}(0) &\rightarrow \{P(\hat{f}(0,u(0))) \vdash P(\hat{f}(0,u(0)))\} \\
\mathrm{cl}^{\psi,\Omega}(k+1) &\rightarrow \mathrm{cl}^{\psi,\Omega}(k) \oplus \{P(u(k+1)) \vdash P(u(k+1))\} \oplus \\
&\qquad (\{P(\hat{f}(k,u(k+1))) \vdash\} \otimes \\
&\qquad \{\vdash P(\hat{f}(k+1,u(k+1)))\})
\end{aligned}$$

Characteristic Clause Set Schema (ctd.)

- The sequence of $\mathrm{CL}(\Psi)\!\downarrow_0, \mathrm{CL}(\Psi)\!\downarrow_1, \mathrm{CL}(\Psi)\!\downarrow_2, \ldots$ is:

$$\{P(u_0) \vdash P(u_0)\ ;\ \vdash P(c)\ ;\ P(c) \vdash\},$$

$$\{P(u_0) \vdash P(u_0)\ ;\ P(f(u_1)) \vdash P(f(u_1))\ ;\ P(u_1) \vdash P(f(u_1))\ ; \\ \vdash P(c)\ ;\ P(f(c)) \vdash\},$$

$$\{P(u_0) \vdash P(u_0)\ ;\ P(f(u_1)) \vdash P(f(u_1))\ ;\ P(f(f(u_2))) \vdash P(f(f(u_2)))\ ; \\ P(u_1) \vdash P(f(u_1))\ ;\ P(f(u_2)) \vdash P(f(f(u_2)))\ ; \\ \vdash P(c)\ ;\ P(f(f(c))) \vdash\}, \ldots$$

- $\mathrm{CL}(\Psi)\!\downarrow_\gamma$ boils down to $\{P(u_1) \vdash P(f(u_1)); \vdash P(c); P(f^\gamma(c)) \vdash\}$.

Projections

- $PR(\Psi)\downarrow_0 : \Bigg\{$

$$\dfrac{\dfrac{\dfrac{\dfrac{P(c) \vdash P(c)}{P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(c), P(g(0,c))}\; w\colon l,r}{P(c) \Rightarrow P(g(0,c)) \vdash P(c), P(c) \Rightarrow P(g(0,c))}\;\Rightarrow\colon r}{\vdash P(c), (P(c) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))}\;\Rightarrow\colon r}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(c), (P(c) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))}\; w\colon l$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(c) \vdash P(c) \qquad P(g(0,c)) \vdash P(g(0,c))}{P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(g(0,c))}\;\Rightarrow\colon l}{P(c), P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(g(0,c))}\; w\colon l}{P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(c) \Rightarrow P(g(0,c))}\;\Rightarrow\colon r}{P(c) \vdash (P(c) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))}\;\Rightarrow\colon r}{P(c), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash (P(c) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))}\; w\colon l$$

$$\dfrac{P(u_0) \vdash P(u_0)}{(\forall x)(P(x) \Rightarrow P(f(x))), P(u_0) \vdash P(u_0), (P(c) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))}\; w\colon l,r \quad \Bigg\}$$

Key Results About *CERES*$_s$

Proposition (Commutativity)

For all $\gamma \in \mathbb{N}$*:*

- $\mathrm{CL}(\Psi \downarrow_\gamma) = \mathrm{CL}(\Psi) \downarrow_\gamma$,
- $PR(\Psi \downarrow_\gamma) = PR(\Psi) \downarrow_\gamma$.

Proposition (Unsatisfiability)

$\mathrm{CL}(\Psi) \downarrow_\gamma$ *is unsatisfiable for all* $\gamma \in \mathbb{N}$ *(i.e.* $\mathrm{CL}(\Psi)$ *is unsatisfiable).*

Proposition (Correctness)

Let $\gamma \in \mathbb{N}$*, then for every clause* $C \in \mathrm{CL}(\Psi) \downarrow_\gamma$ *there exists an* $\mathbf{LK}_s$*-proof* $\pi \in PR(\Psi) \downarrow_\gamma$ *with end-sequent* $C \circ S(\gamma)$.

A Refutation Schema

- Let $R = (\varrho, \delta)$ where
  - $\varrho(0, u) \rightarrow r(\delta(0, u);\ P(\hat{f}(0, c)) \vdash;\ P(\hat{f}(0, c)))$,
  - $\varrho(k + 1, u) \rightarrow r(\delta(k + 1, u);\ P(\hat{f}(k + 1, c)) \vdash;\ P(\hat{f}(k + 1, c)))$,
  - $\delta(0, u) \rightarrow \vdash P(c)$,
  - $\delta(k + 1, u) \rightarrow r(\delta(k, u);\ P(u(k{+}1)) \vdash P(f(u(k{+}1)));\ P(\hat{f}(k, c)))$.
- Let $\hat{pre} : \omega \rightarrow \omega$ be a defined function symbol, then define the function $\hat{pre}(n)$ as:
  - $\hat{pre}(0) \rightarrow 0$, and
  - $\hat{pre}(k + 1) \rightarrow k$.
- $\theta = \{u \leftarrow \lambda k.\hat{f}(\hat{pre}(k), c)\}$.

A Refutation Schema (ctd.)

- $R\theta\downarrow_\gamma$ is a resolution refutation for all $\gamma \in \mathbb{N}$:

$$\begin{array}{rcl} R\theta\downarrow_0 & = & r(\vdash P(c)\ ;\ P(c) \vdash\ ;\ P(c)) \\ R\theta\downarrow_1 & = & r(\ r(\vdash P(c)\ ;\ P(c) \vdash P(f(c))\ ;\ P(c)); \\ & & \quad P(f(c)) \vdash\ ;\ P(f(c))) \\ R\theta\downarrow_2 & = & r(\ r(\ r(\vdash P(c)\ ;\ P(c) \vdash P(f(c))\ ;\ P(c)); \\ & & \qquad P(f(c)) \vdash P(f(f(c)))\ ;\ P(f(c))); \\ & & \quad P(f(f(f))) \vdash\ ;\ P(f(f(c)))) \\ & \vdots & \end{array}$$

Refutation to $\mathbf{LK}_s$-skeleton

Let ϱ be a normalized resolution refutation. Then the transformation $TR(\varrho)$ is defined inductively:

- if $\varrho = C$ for a clause C, then $TR(\varrho) = C$,
- if $\varrho = r(\varrho_1; \varrho_2; P)$, then $TR(\varrho)$ is:

$$\dfrac{\dfrac{\begin{array}{c}(TR(\varrho_1))\\ \Gamma \vdash \Delta, P, \ldots, P\end{array}}{\Gamma \vdash \Delta, P}\ c\colon r* \qquad \dfrac{\begin{array}{c}(TR(\varrho_2))\\ P, \ldots, P, \Pi \vdash \Lambda\end{array}}{P, \Pi \vdash \Lambda}\ c\colon l*}{\Gamma, \Pi \vdash \Delta, \Lambda}\ cut$$

An Example (ctd.)

- Let $A = (\forall x)(P(x) \Rightarrow P(f(x)))$ and $B = (P(c) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))$, then:

$$\cfrac{\vdash P(c) \qquad P(c) \vdash}{\vdash}\ cut$$

An Example (ctd.)

- Let $A = (\forall x)(P(x) \Rightarrow P(f(x)))$ and $B = (P(c) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))$, then:

$$\dfrac{\vdash P(c) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(c) \vdash P(c) \qquad P(g(0,c)) \vdash P(g(0,c))}{P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(g(0,c))}\ \Rightarrow: l}{P(c), P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(g(0,c))}\ w: l}{P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(c) \Rightarrow P(g(0,c))}\ \Rightarrow: r}{P(c) \vdash B}\ \Rightarrow: r}{P(c), A \vdash B}\ w: l}{A \vdash B}\ cut$$

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$$
\begin{prooftree}
\AxiomC{$P(c) \vdash P(c)$}
\doubleLine
\RightLabel{$w\colon l, r$}
\UnaryInfC{$P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(c), P(g(0,c))$}
\RightLabel{$\Rightarrow\colon r$}
\UnaryInfC{$P(c) \Rightarrow P(g(0,c)) \vdash P(c), P(c) \Rightarrow P(g(0,c))$}
\RightLabel{$\Rightarrow\colon r$}
\UnaryInfC{$\vdash P(c), B$}
\RightLabel{$w\colon l$}
\UnaryInfC{$A \vdash P(c), B$}
\AxiomC{$P(c) \vdash P(c)$}
\AxiomC{$P(g(0,c)) \vdash P(g(0,c))$}
\RightLabel{$\Rightarrow\colon l$}
\BinaryInfC{$P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(g(0,c))$}
\RightLabel{$w\colon l$}
\UnaryInfC{$P(c), P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(g(0,c))$}
\RightLabel{$\Rightarrow\colon r$}
\UnaryInfC{$P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(c) \Rightarrow P(g(0,c))$}
\RightLabel{$\Rightarrow\colon r$}
\UnaryInfC{$P(c) \vdash B$}
\RightLabel{$w\colon l$}
\UnaryInfC{$P(c), A \vdash B$}
\RightLabel{cut}
\BinaryInfC{$A, A \vdash B, B$}
\end{prooftree}
$$

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\UnaryInfC{$\vdash P(c), B$}
\RightLabel{$w\colon l$}
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\AxiomC{$P(c) \vdash P(c)$}
\AxiomC{$P(g(0,c)) \vdash P(g(0,c))$}
\RightLabel{$\Rightarrow\colon l$}
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\RightLabel{$w\colon l$}
\UnaryInfC{$P(c), P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(g(0,c))$}
\RightLabel{$\Rightarrow\colon r$}
\UnaryInfC{$P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(c) \Rightarrow P(g(0,c))$}
\RightLabel{$\Rightarrow\colon r$}
\UnaryInfC{$P(c) \vdash B$}
\RightLabel{$w\colon l$}
\UnaryInfC{$P(c), A \vdash B$}
\RightLabel{cut}
\BinaryInfC{$A, A \vdash B, B$}
\doubleLine
\RightLabel{$c\colon l, r$}
\UnaryInfC{$(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (P(c) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))$}
\end{prooftree}
$$

Main Theorem

Theorem (ACNF)

Let Ψ *be a proof schema with end-sequent* $S(n)$, *and let* R *be a resolution refutation schema of* $\mathrm{CL}(\Psi)$. *Then for all* $\alpha \in \mathbb{N}$ *there exists a normalized* $\mathbf{LK}_s$*-proof* π *of* $S(\alpha)$ *with at most atomic cuts such that its size* $l(\pi)$ *is polynomial in* $l(R{\downarrow_\alpha}) \cdot l(PR(\Psi){\downarrow_\alpha})$.

- Drawback: the method is inherently incomplete.

Summary

Whole CERES_s Procedure

- Phase 1 of CERES_s (schematic construction):
  - compute $\mathrm{CL}(\Psi)$;
  - compute $\mathrm{PR}(\Psi)$;
  - construct a resolution refutation schema $\mathcal{R}$ of $\mathrm{CL}(\Psi)$ and a substitution schema ϑ.
  - then ACNF schema is $(\mathrm{PR}(\Psi), \mathcal{R}, \vartheta)$.
- Phase 2 of CERES_s (evaluation, given a number α):
  - compute $\mathrm{PR}(\Psi)\downarrow_\alpha$;
  - compute $\mathcal{R}\vartheta\downarrow_\alpha$ and $T_\alpha : \mathit{TR}(\mathcal{R}\vartheta\downarrow_\alpha)$;
  - append the corresponding projections in $\mathrm{PR}(\Psi)\downarrow_\alpha$ to T_α, propagate the contexts down and append necessary contractions at the end of the proof.

Future Work

- Extract valuable information such as Herbrand sequent from the ACNF schema.
- Investigate the resolution calculus (paramodulation, decidable fragments, etc.).
- Extend proof schema systems and the method to multiple parameters.

Questions?