Intensionalisation of Logical Operators

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Semantics based on an assertibility relation

C is a set of possible worlds (a context).

$C \Vdash \bot$ iff $C = \emptyset$.

$C \Vdash p$ iff for all $v \in C$, $v(p) = 1$.

$C \Vdash \varphi \wedge \psi$ iff $C \Vdash \varphi$ and $C \Vdash \psi$.

$C \Vdash \varphi \vee \psi$ iff for some D, E, $D \cup E = C$, $D \Vdash \varphi$ and $E \Vdash \psi$.

$C \Vdash \varphi \rightarrow \psi$ iff $D \Vdash \psi$ for all $D \subseteq C$ such that $D \Vdash \varphi$.

$$\neg\varphi =_{Df} \varphi \rightarrow \bot$$

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Consequence relation

Definition

$\Delta \vDash \psi$ iff for all C, if $C \Vdash \Delta$, then $C \Vdash \psi$.

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$\Delta \vDash \psi$ iff for all C, if $C \Vdash \Delta$, then $C \Vdash \psi$.

Fact

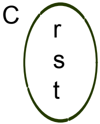
$\vDash$ *is identical with the consequence relation of classical logic.*

"Extensional" principle for disjunction

If $C \Vdash \varphi$ and $D \Vdash \psi$, then $C \cup D \Vdash \varphi \vee \psi$.

Factual sentences

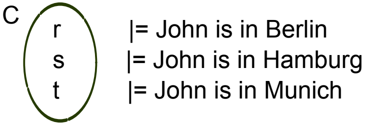
$C \Vdash$ John is in Germany.



|= John is in Berlin
|= John is in Hamburg
|= John is in Munich

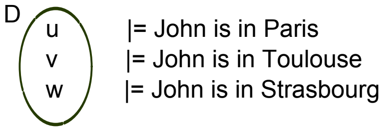
Factual sentences

$C \Vdash$ John is in Germany.



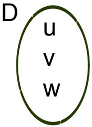
Factual sentences

$D \Vdash$ John is in France.



Factual sentences

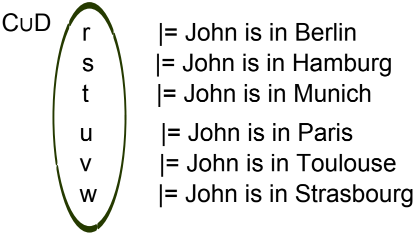
$D \Vdash$ John is in France.



$\models$ John is in Paris
$\models$ John is in Toulouse
$\models$ John is in Strasbourg

Factual sentences

$C \cup D \Vdash$ John is in Germany or he is in France.



Factual sentences

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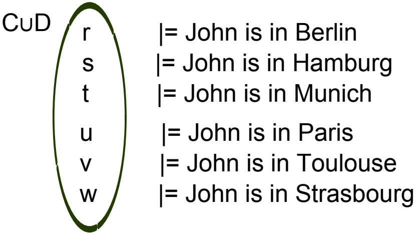
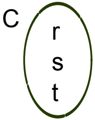


Contextual sentences

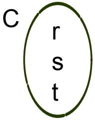
$C \Vdash$ All suspects are men.



|= John commited the crime
|= Robert commited the crime
|= Michael commited the crime

Contextual sentences

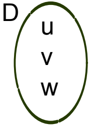
$C \Vdash$ All suspects are men.



|= John commited the crime
|= Robert commited the crime
|= Michael commited the crime

Contextual sentences

$D \Vdash$ All suspects are women.



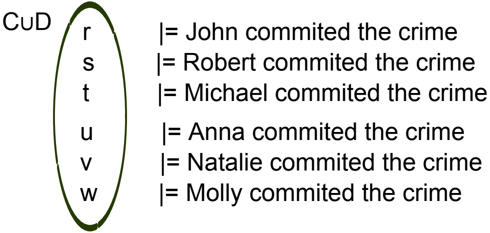
$\models$ Anna commited the crime
$\models$ Natalie commited the crime
$\models$ Molly commited the crime

Contextual sentences

$D \Vdash$ All suspects are women.

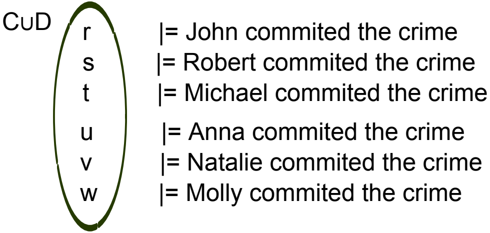
Contextual sentences

$C \cup D \not\Vdash$ All suspects are men or all suspects are women.



Contextual sentences

$C \cup D \nVdash$ All suspects are men or all suspects are women.



Strict disjunction

$C \Vdash \varphi \vee \psi$ iff $C \Vdash \varphi$ or $C \Vdash \psi$.

Inquisitive semantics (J. Groenendijk)

$C \Vdash \bot$ iff $C = \emptyset$.

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$$\neg\varphi =_{Df} \varphi \rightarrow \bot$$

Semantics of assertibility

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$$\neg\varphi =_{Df} \varphi \rightarrow \bot$$

Negation and implication

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

Paul Grice — Denial of a conditional

Sometimes a denial of a conditional has the effect of a refusal to assert the conditional in question, characteristically because the denier does not think that there are adequate non-truth-functional grounds for such an assertion.

(Paul Grice, Indicative conditionals)

Weak negation
expressing a refusal to assert a sentence

$C \Vdash {\sim}\varphi$ iff $C \nVdash \varphi$.

Semantics of assertibility with weak negation

For every C, $C \nVdash \bot$.

$C \Vdash p$ iff for all $v \in C$, $v(p) = 1$.

$C \Vdash {\sim}\varphi$ iff $C \nVdash \varphi$.

$C \Vdash \varphi \wedge \psi$ iff $C \Vdash \varphi$ and $C \Vdash \psi$.

$C \Vdash \varphi \vee \psi$ iff $C \Vdash \varphi$ or $C \Vdash \psi$.

$C \Vdash \varphi \rightarrow \psi$ iff $D \Vdash \psi$ for all nonempty $D \subseteq C$ such that $D \Vdash \varphi$.

$$\neg\varphi =_{Df} \varphi \rightarrow \bot$$

Two kinds of modal operators

$$\Box\varphi =_{Df} \neg\neg\varphi, \qquad \Diamond\varphi =_{Df} \sim\Box\sim\varphi.$$

$$\blacksquare\varphi =_{Df} \neg\sim\varphi, \qquad \blacklozenge\varphi =_{Df} \sim\blacksquare\sim\varphi.$$

The relationships between the modalities

Fact

(i) $\Box\varphi \equiv \blacksquare\blacklozenge\varphi$,

(ii) $\Diamond\varphi \equiv \blacklozenge\blacksquare\varphi$.

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Proof.

(i) $\Box\varphi = \neg\neg\varphi \equiv \neg{\sim}{\sim}\neg{\sim}{\sim}\varphi = \blacksquare\blacklozenge\varphi$.

(ii) $\Diamond\varphi = {\sim}\neg\neg{\sim}\varphi \equiv {\sim}\neg{\sim}{\sim}\neg{\sim}\varphi = \blacklozenge\blacksquare\varphi$.

□

Semantics of the modal operators

Fact

(i) $\Box\varphi$ *is assertible in* C *iff* φ *is (classically) true in every world of* C.

(ii) $\blacksquare\varphi$ *is assertible in* C *iff* φ *is assertible in every subcontext of* C.

(iii) $\Diamond\varphi$ *is assertible in* C *iff* φ *is (classically) true in some world of* C.

(iv) $\blacklozenge\varphi$ *is assertible in* C *iff* φ *is assertible in some subcontext of* C.

Two dual operators: $\oplus$ and $\otimes$

$$\varphi_1 \oplus \ldots \oplus \varphi_n =_{Df} \Box(\varphi_1 \vee \ldots \vee \varphi_n) \wedge (\Diamond\varphi_1 \wedge \ldots \wedge \Diamond\varphi_n).$$

$$\varphi_1 \otimes \ldots \otimes \varphi_n =_{Df} \Diamond(\varphi_1 \wedge \ldots \wedge \varphi_n) \vee (\Box\varphi_1 \vee \ldots \vee \Box\varphi_n).$$

Two dual operators: $\oplus$ and $\otimes$

$$\varphi_1 \oplus \ldots \oplus \varphi_n =_{Df} \Box(\varphi_1 \vee \ldots \vee \varphi_n) \wedge (\Diamond\varphi_1 \wedge \ldots \wedge \Diamond\varphi_n).$$

$$\varphi_1 \otimes \ldots \otimes \varphi_n =_{Df} \Diamond(\varphi_1 \wedge \ldots \wedge \varphi_n) \vee (\Box\varphi_1 \vee \ldots \vee \Box\varphi_n).$$

Semantics of $\oplus$

$C \Vdash \varphi_1 \oplus \ldots \oplus \varphi_n$.

Every disjunct is true in at least one possible world and in every possible world at least one disjunct is true.

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Fact

(i) $\Diamond\varphi_1 \wedge \ldots \wedge \Diamond\varphi_n \equiv \blacklozenge(\varphi_1 \oplus \ldots \oplus \varphi_n)$,

(ii) $\Diamond\varphi_1 \vee \ldots \vee \Diamond\varphi_n \equiv \blacklozenge(\varphi_1 \otimes \ldots \otimes \varphi_n)$,

(iii) $\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n \equiv \blacksquare(\varphi_1 \oplus \ldots \oplus \varphi_n)$,

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(iv) $\Box\varphi_1 \vee \ldots \vee \Box\varphi_n \equiv \blacksquare(\varphi_1 \otimes \ldots \otimes \varphi_n)$.

Conditional proof

$\Diamond\neg p, p \vDash \bot$

but $\Diamond\neg p \nvDash p \rightarrow \bot$

i.e. $\Diamond\neg p \nvDash \neg p$.

Conditional proof

$\Diamond\neg p, p \vDash \bot$

but $\Diamond\neg p \nvDash p \to \bot$

i.e. $\Diamond\neg p \nvDash \neg p$.

Conditional proof

$$\Diamond\neg p, p \vDash \bot$$

but $\Diamond\neg p \nvDash p \rightarrow \bot$

i.e. $\Diamond\neg p \nvDash \neg p$.

Conditional proof

$\Diamond\neg p, p \vDash \bot$

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Restricted conditional proof $(\varphi : \psi)/\varphi \to \psi$

In the scope of a hypotetical assumption, not all formulas from the outer proof are available.

We can use only $\sim$-free formulas and formulas of the form $\varphi \to \psi$.

A system of natural deduction

$(\wedge I)$	$\varphi, \psi / \varphi \wedge \psi$	$(\wedge E)$	(i) $\varphi \wedge \psi / \varphi$, (ii) $\varphi \wedge \psi / \psi$
$(\vee I)$	(i) $\varphi / \varphi \vee \psi$, (ii) $\psi / \varphi \vee \psi$	$(\vee E)$	$\varphi \vee \psi, [\varphi : \chi], [\psi : \chi] / \chi$
$(\rightarrow I)^*$	$(\varphi : \psi) / \varphi \rightarrow \psi$	$(\rightarrow E)$	$\varphi, \varphi \rightarrow \psi / \psi$
$(\bot I)$	$\varphi, \sim \varphi / \bot$	(IP)	$[\sim \varphi : \bot] / \varphi$

A system of natural deduction

$$(R1) \quad \Box p \, / \, p$$

$$(R2) \quad / \, \Box(\varphi \vee \neg\varphi),$$

$$(R3) \quad \Box\varphi \wedge \Box\psi \, / \, \Box(\varphi \wedge \psi)$$

$$(R4) \quad \Diamond\varphi_1 \wedge \ldots \wedge \Diamond\varphi_n / \blacklozenge(\varphi_1 \oplus \ldots \oplus \varphi_n).$$

Theorem

The system of natural deduction is sound and complete with respect to the semantics of assertibility with weak negation.

$\sim(p \to q) \nvdash p$

1	$\sim(p \to q)$	premise
2	$\sim p$	hyp. assumption
3	p	hyp. assumption
4	$\bot$	2,3 $(\bot I)$
5	q	4 Ex falso quodlibet (derivable rule)
6	$p \to q$	3-5 $(\to I)^*$!!!!!!!
7	$\bot$	1,6 $(\bot I)$
8	p	2-7 (IP)