Levels of Knowledge and Belief

TbiLLC 2013

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September 23, 2013

An epistemic situation

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Ann and Bob are playing poker. Ann is having an Ace and of course she knows that.There is a mirror behind Ann, so Bob sees that Ann has an Ace but she does not know that he knows. However since Bob is notorious for cheating she does consider possible that he knows.

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Axioms:

- Truth: $K_i\varphi \rightarrow \varphi$
- Positive Introspection: $K_i\varphi \rightarrow K_iK_i\varphi$
- Negative Intropsection $\neg K_i\varphi \rightarrow K_i\neg K_i\varphi$

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Examples

- Card Game
- Security
- Knowledge of Rationality in GT

Levels of Knowledge

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More generaly: a level of knowledge is a maximally consistent subset of the language $\mathcal{L}_\varphi$ generated by $\varphi, \vee, \wedge, \neg, K_1 \ldots K_n$

But. . .

In general we are only interested in a *fragment* of the language.

- Positive Knowledge: Only $K_i \ldots K_j\varphi$
- Limited Reasoning: At most 5 nested knowledge operators
- Tractability
- . . .

Thus...

Official Definition: Let $\mathcal{F}$ be a subset of the formulae generated by $\varphi, \vee, \wedge, \neg, K_1 \ldots K_n$. (fragment of interest)
Then a level of $\mathcal{F}$-knowledge of φ is a subset L of $\mathcal{F}$ such that $L = \mathcal{F} \cap T$ for some maximally consistent subset T of $\mathcal{L}_\varphi$

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Equivalently $L \subset \mathcal{F}$ is a level of $\mathcal{F}$-knowledge iff there is a Kripke Model $(\mathcal{M}, s)$ such that

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Notation: $(\mathcal{M}, s)$ realizes L.

Our Main Questions

- How does the expressive power of levels of $\mathcal{F}$-knowledge depend on $\mathcal{F}$
- Which levels are realizable in *finite* Kripke Models
- How Do levels of knowledge behave under incoming information?

Measure of Expressive Power

- Expressive Power: Number of different situations that can be distinguished by $\mathcal{F}$.
- Levels of knowledge are subsets of $\mathcal{F}$. If $\mathcal{F}$ is infinite, there are uncountably many subsets of $\mathcal{F}$.
- How many levels are there (countable vs. uncountable)

Results I

For the following fragments there are only countably many levels of knowledge:

- $\mathcal{F}_K$ generated by $K_1, \ldots, K_n, \varphi$ (Parikh,Krasucki)

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- $\mathcal{F}_K$ generated by $K_1, \ldots, K_n, \varphi$ (Parikh,Krasucki)
- $\mathcal{F}_L$ generated by $L_1, \ldots, L_n, \varphi$ (L dual of K)
- $\mathcal{F}_D$ generated by $D_I : I \subseteq \{1 \ldots n\}, \varphi$ where $D_I = \bigvee_{i \in I} K_i$
- $\mathcal{F}_\wedge$ generated by $K_1, \ldots, K_n, \wedge, \varphi$
- open, but very close partial results: The language generated by $K_1, \ldots, K_n, \vee, \varphi$
  solved for 2 agents, bounded number of $\vee$

Results II

For the following fragments there are uncountably many levels knowledge (for $n \geq 2$ agents):

- $\mathcal{F}_{\neg}$ generated by $K_1, \dots K_n, \neg, \varphi$
- $\mathcal{F}_{L,K}$ generated by $K_1, \dots, K_n, L_1, \dots, L_n, \varphi$
- $\mathcal{F}_J$ generated by $J_1, \dots, J_n, x$, where $J_i x$ is defined as $J_i x := K_i x \vee K_i \neg\varphi$ (knowing whether) (Heifetz et al)

General Lessons

- Negation increases expressive power
- $\vee$ and $\wedge$: no effect on expressive power
- Tools developed work for many other modal logics (BQO-theory)
- In the belief case: Already the subset generated by $B_1 \dots B_n, \varphi$ has uncountably many types (Parikh, Pacuit)
  $\Rightarrow$ Finer discriminants needed
- T axiom is crucial for countability results.

Representability in Kripke models

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Theorem

For the fragments $\mathcal{F}$ above that only have countably many levels of $\mathcal{F}$-knowledge every level of knowledge is realized in a finite Kripke Model.

Levels and Dynamics

Question: How do levels of information behave under $\oplus$-updates?

$$\langle \varphi, K_A\varphi, K_B\varphi \rangle \stackrel{\oplus-\text{update}}{\Rightarrow} \langle \varphi, K_A\varphi, K_B\varphi, K_A K_B\varphi \rangle$$

Levels and Dynamics

Question: How do levels of information behave under $\oplus$-updates?

Answer for positive knowledge, i.e. $\mathcal{F}$ generated by $K_1, \ldots, K_n, \varphi$.

Theorem Let $L_1 \neq \emptyset$ and L_2 be levels of positive knowledge. Let $\mathcal{M}, s$ be a Kripke model realizing L_1. Then there is an event model $\mathcal{E}$ and a Kripke model $\mathcal{L}, t$ realizing L_2 with $\mathcal{M}, s \oplus \mathcal{E} = \mathcal{L}, t$ if and only if $L_1 \subseteq L_2$.

Conclusion

- Complexity of levels of $\mathcal{F}$-knowledge depends crucially upon $\mathcal{F}$
- T axiom reduces complexity
- $\neg$ increases expressive power (also for the belief case)
- $\wedge, \vee$ do not add (too much) expressive power
- Little expressive power: Representable in finite models
- Connection between levels of knowledge and dynamics

Future Work

- Understand how levels are related

  Given the level of p and q, what can be said about the level of $p \wedge q$...

  That is: Every Kripke strucute defines a map from the Lindenbaum Algebra to the set of levels:

  $$\Psi : \mathcal{T} \to \{Lev\}$$

  characterize this map
- Finer discriminants then cardinality
- General modal logics

Thank You!