Analytic calculi for non-classical logics: the Baha'i method

Agata Ciabattoni

Vienna University of Technology (TU Vienna)

Non-classical logics

Non-classical logics provide languages for reasoning, e.g., about dynamic data structures, resources, algebraic structures, vague or inconsistent information . . .

They are often described/introduced by adding suitable properties to known systems:

- Hilbert axioms
- Semantic conditions

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- Hilbert axioms
- Semantic conditions

Example: Gödel logic is obtained from intuitionistic logic

- by adding the Hilbert axiom $(\phi \to \psi) \vee (\psi \to \phi)$, or
- by imposing on intuitionistic frames the strong connectedness of the accessibility relation $\leqslant$, i.e. $\forall x \forall y \forall z((x \leqslant y \wedge x \leqslant z) \to (y \leqslant z \vee z \leqslant y))$.

Analytic calculi

The applicability/usefulness of non-classical logics strongly depends on the availability of analytic calculi.

Analytic calculi are

- useful for establishing various properties of logics
- key for developing automated reasoning methods.

Sequent Calculus

(Gentzen 1934)

Sequents

$$A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$$

Axioms

E.g., $A \Rightarrow A$

Rules

Logical, Structural and

$$\frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi} \; Cut$$

Sequent Calculus – state of the art

+ Cut-free sequent calculi have been successfully used
  - to prove decidability, interpolation, consistency, ...
  - to give syntactic proofs of algebraic properties for which (in particular cases) semantic methods are not known or do not work well

- Many useful and interesting logics have no cut-free sequent calculus

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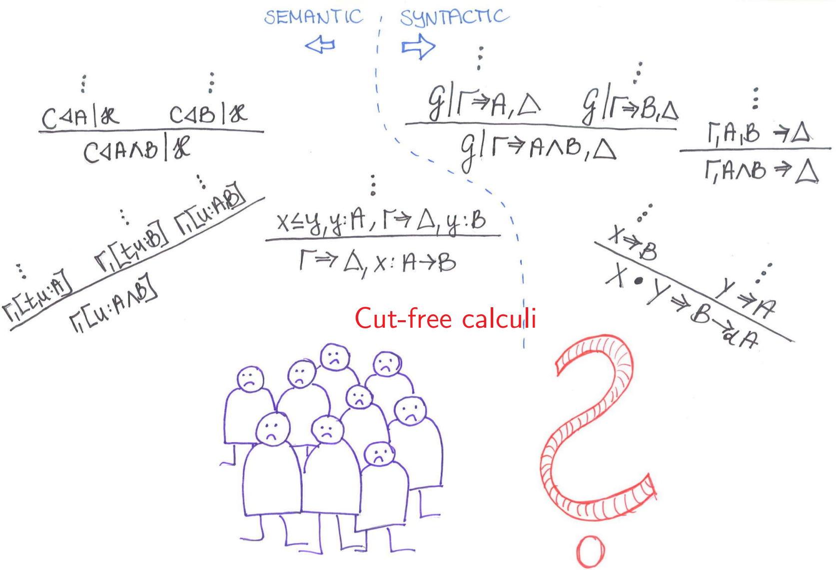
- Many useful and interesting logics have no cut-free sequent calculus

A large range of extensions of sequent calculus have been introduced



Cut-free calculi

Extensions of the sequent calculus

Syntactic Formalisms

- hypersequent calculus (Avron, TU Vienna, . . .)
- display calculus (Belnap, Wansing, Goré, . . .)
- nested sequents (Guglielmi, Brünnler, Fitting, . . .)
- ...

Semantic Formalisms

- labelled systems (Gabbay, Negri, Viganó, . . .)
- many placed sequents (TU Vienna) – finite valued logics
- sequents of relations (TU Vienna) – many-valued logics
- ...

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Long standing dispute?!

State of the art

The definition of analytic calculi is usually logic-tailored.

Steps:

(i) choosing a framework

(ii) looking for the "right" inference rule(s)

(iii) proving cut-elimination

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Our Dream

- Define analytic calculi for non-classical logics in a systematic and algorithmic way
- Characterize the expressive power of the various frameworks

The Baha'i method

The Baha'i method

A general method to introduce analytic calculi for large classes of logics which works for syntactic and semantic formalisms and provide a unifying perspective of them.

(with N. Galatos and K. Terui)

The Baha'i method

The Baha'i faith believes in the unity of all religion. One God and many prophets: Buddha, Krishna, Jesus, Mohammed, . . . (cf. Lonely Planet)

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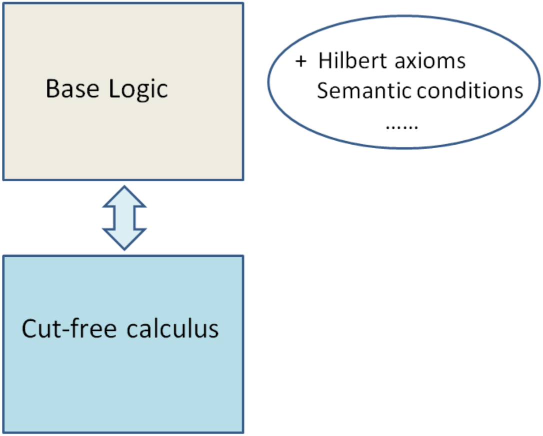
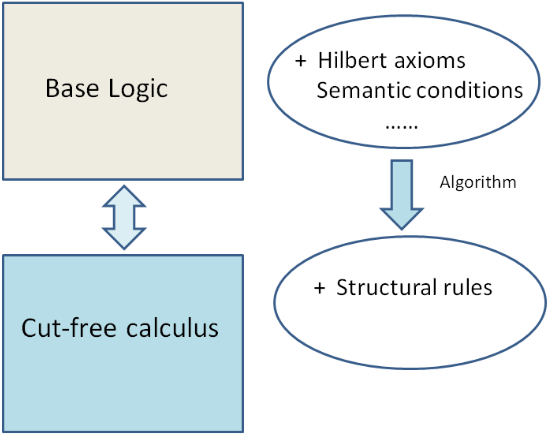


The Baha'i method



The Baha'i method – Steps

(Step 1): Classification of the properties (axioms, frame conditions ...)
(Step 2): Transformation procedure: from conditions to structural rules that preserve cut-elimination when added to the base calculus
(Step 3): Completion of the generated rules (when needed)

Step 1: classification

The formulas (Hilbert axioms, semantics conditions..) are **classified** according to the

polarity of their connectives/quantifiers

(J.-M. Andreoli, 1992)

- *Positive* polarity: rule introducing the connective/quantifier on the *left* is invertible
- *Negative* polarity: rule introducing the connective/quantifier on the *right* is invertible

Step 2: transformation procedure

Given a base cut-free calculus $\mathcal{C}$. The algorithm is based on:

Ingredient 1

The use of the invertible rules of $\mathcal{C}$

Ingredient 2: Ackermann Lemma

An algebraic equation $t \leq u$ is equivalent to a quasiequation $u \leq x \Longrightarrow t \leq x$, and also to $x \leq t \Longrightarrow x \leq u$, where x is a fresh variable not occurring in t, u.

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Example: the sequent $A \vdash B$ is equivalent to

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash B, \Delta} \qquad\qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \vdash \Delta}$$

(Γ, Δ fresh metavariables for lists of formulas)

Intermediate logics: a case study

Logics between intuitionistic and classical logic

Semantic approach

- imposing on intuitionistic Kripke frames additional conditions on the (transitive and reflexive) accessibility relation $\leqslant$

Syntactic approach

- extending intuitionistic propositional calculus with Hilbert axioms

$\mathbf{LJ}$ + axioms/frame conditions = no cut-elimination!

The Bahai's method at work

- hypersequent calculus
- labelled sequent calculus
- display calculus

(-, N. Galatos and K. Terui). *Annals of Pure and Applied Logic*, 2012.

(-, N. Galatos and K. Terui). *LICS 2008*.

Hypersequent calculus

(Avron'87)

It is obtained by embedding sequents into hypersequents

$$\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$$

where all $\Gamma_i \Rightarrow \Pi_i$ are ordinary sequents (components).

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E.g. Rules

$$\frac{G|\Gamma \Rightarrow A \quad G|A, \Delta \Rightarrow \Pi}{G|\Gamma, \Delta \Rightarrow \Pi}\ Cut \qquad \frac{}{G|A \Rightarrow A}\ Identity$$

$$\frac{G|\Gamma \Rightarrow A \quad G|B, \Delta \Rightarrow \Pi}{G|\Gamma, A \rightarrow B, \Delta \Rightarrow \Pi} \rightarrow l \qquad \frac{G|A, \Gamma \Rightarrow B}{G|\Gamma \Rightarrow A \rightarrow B} \rightarrow r$$

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$$\frac{G|\Gamma \Rightarrow A \quad G|B, \Delta \Rightarrow \Pi}{G|\Gamma, A \to B, \Delta \Rightarrow \Pi} \to l \qquad \frac{G|A, \Gamma \Rightarrow B}{G|\Gamma \Rightarrow A \to B} \to r$$

and, to manipulate the additional layer of structure,

$$\frac{G}{G \mid \Gamma \Rightarrow A}\ (\mathrm{ew}) \qquad \frac{G \mid \Gamma' \Rightarrow B \mid \Gamma \Rightarrow A \mid G'}{G \mid \Gamma \Rightarrow A \mid \Gamma' \Rightarrow B \mid G'}\ (\mathrm{ee}) \qquad \frac{G \mid \Gamma \Rightarrow A \mid \Gamma \Rightarrow A}{G \mid \Gamma \Rightarrow A}\ (\mathrm{ec})$$

Structural rules in the hypersequent calculus

Example: Gödel logic $(= \mathsf{IL} + (\phi \to \psi) \vee (\psi \to \phi))$

$$\frac{G \mid \Delta_1, \Gamma_1 \Rightarrow \Pi_1 \quad G \mid \Delta_2, \Gamma_2 \Rightarrow \Pi_2}{G \mid \Delta_2, \Gamma_1 \Rightarrow \Pi_1 \mid \Delta_1, \Gamma_2 \Rightarrow \Pi_2} \ (com)$$

(Avron '91)

The Baha'i method for hypersequent calculus

- hypersequent calculi

  For intermediate logics

  - Base system: hypersequent version of Gentzen LJ calculus
  - Properties of intermediate logics: Hilbert axioms
  - Generated Rules: structural hypersequent rules
- labelled sequent calculi
- display calculi

The Baha'i method for hypersequent calculus

(Step 1): Classification of axioms based on the polarity of connectives/quantifiers

(Step 2): Transformation procedure

(Step 3): Completion of the generated rules

From axioms to hypersequent rules

Definition (Classification based on LJ)

The classes $\mathcal{P}_n, \mathcal{N}_n$ of positive and negative axioms are:

- $\mathcal{P}_0 ::= \mathcal{N}_0 ::=$ atomic formulas
- $\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \wedge \mathcal{P}_{n+1} \mid \bot$
- $\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{P}_{n+1} \to \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid \top$

$$\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[r] \arrow[dr] & \mathcal{P}_2 \arrow[r] \arrow[dr] & \mathcal{P}_3 \arrow[r] \arrow[dr] & \mathcal{P}_4 \arrow[r] \arrow[dr] & \cdots \\
\mathcal{N}_0 \arrow[r] \arrow[ur] & \mathcal{N}_1 \arrow[r] \arrow[ur] & \mathcal{N}_2 \arrow[r] \arrow[ur] & \mathcal{N}_3 \arrow[r] \arrow[ur] & \mathcal{N}_4 \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}$$

Our Result

Theorem

Any axiom within the class $\mathcal{P}_3$ can be transformed into equivalent structural hypersequent rules that preserve cut-elimination once added to LJ.

Some examples

- $\sim \phi \vee \phi$
- $(\phi \rightarrow \psi) \vee (\psi \rightarrow \phi)$
- $\sim \phi \vee \sim\sim \phi$
- $\phi_0 \vee (\phi_0 \rightarrow \phi_1) \vee \cdots \vee (\phi_0 \wedge \cdots \wedge \phi_{k-1} \rightarrow \phi_k)$
- $\bigvee_{i=0}^{k} (\phi_i \rightarrow \bigvee_{j \neq i} \phi_j)$

From axioms to hypersequent rules II

Example (Jankov logic: Axiom $\sim \phi \vee \sim\sim \phi$)

Equivalent to $$\frac{}{G \mid \Rightarrow \sim \phi \mid \Rightarrow \sim\sim \phi}$$

Invertibility $$\frac{}{G \mid \phi \Rightarrow \mid \sim \phi \Rightarrow}$$

Ackermann Lemma 1 $$\frac{G \mid \Gamma' \Rightarrow \sim \phi}{G \mid \phi \Rightarrow \mid \Gamma' \Rightarrow}$$

Invertibility $$\frac{G \mid \Gamma', \phi \Rightarrow}{G \mid \phi \Rightarrow \mid \Gamma' \Rightarrow}$$

Ackermann Lemma 2 $$\frac{G \mid \Gamma \Rightarrow \phi \qquad G \mid \Gamma', \phi \Rightarrow}{G \mid \Gamma \Rightarrow \mid \Gamma' \Rightarrow}$$

Equivalent rule $$\frac{G \mid \Gamma, \Gamma' \Rightarrow}{G \mid \Gamma \Rightarrow \mid \Gamma' \Rightarrow}$$

Automated approach: *Axiomcalc*

Our theorem

Any axiom within the class $\mathcal{P}_3$ can be transformed into equivalent structural hypersequent rules that preserve cut-elimination once added to LJ.

Our system: *Axiomcalc*

http://www.logic.at/people/lara/axiomcalc.html

The Bahai's method at work II

- hypersequent calculus
- labelled sequent calculus
- display calculus

(-, P. Maffezioli and L. Spendier). *Tableaux* 2013.

From frame conditions to labelled calculi

Definition (Semantics)

- Intuitionistic frame $\mathfrak{F} = \langle X, \leqslant \rangle$ where X is a non-empty set and $\leqslant$ is a reflexive and transitive (accessibility) relation on X.
- Intuitionistic model $\mathfrak{M} = \langle \mathfrak{F}, \Vdash \rangle$
- $\Vdash$ (forcing): relation between elements of X and atomic formulas

Example

- *Jankov logic*:
  $\forall x \forall y \forall z((x \leqslant y \wedge x \leqslant z) \to \exists w(y \leqslant w \wedge z \leqslant w))$
- *Gödel logic*: $\forall x \forall y \forall z((x \leqslant y \wedge x \leqslant z) \to (y \leqslant z \vee z \leqslant y))$
- *Bd_2*: $\forall x \forall y \forall z((x \leqslant y \wedge y \leqslant z) \to (y \leqslant x \vee z \leqslant y))$

Labelled calculi for intermediate logics

Labelled calculi (Kanger 1957, ...):

- Extension of standard sequent calculi in which the semantics is explicit part of the syntax
- Each formula α receives a label x, e.g. $x : \alpha$
- Labels occur also in expressions of the (reflexive and transitive) accessibility relation $x \leqslant y$

$x \Vdash \alpha$ (x "forces" α) represented as $x : \alpha$

Labelled calculi for intermediate logics

Labelled calculus *G3I* for intuitionistic logic (Negri):

$$x \leqslant y, x : p, \Gamma \Rightarrow \Delta, y : p$$

$$\frac{x : \alpha, x : \beta, \Gamma \Rightarrow \Delta}{x : \alpha \wedge \beta, \Gamma \Rightarrow \Delta} \ (\wedge, l) \qquad \frac{\Gamma \Rightarrow \Delta, x : \alpha \quad \Gamma \Rightarrow \Delta, x : \beta}{\Gamma \Rightarrow \Delta, x : \alpha \wedge \beta} \ (\wedge, r)$$

$$\frac{x : \alpha, \Gamma \Rightarrow \Delta \quad x : \beta, \Gamma \Rightarrow \Delta}{x : \alpha \vee \beta, \Gamma \Rightarrow \Delta} \ (\vee, l) \qquad \frac{\Gamma \Rightarrow \Delta, x : \alpha, x : \beta}{\Gamma \Rightarrow \Delta, x : \alpha \vee \beta} \ (\vee, r)$$

$$\frac{x \leqslant y, x : \alpha \to \beta, \Gamma \Rightarrow \Delta, y : \alpha \quad x \leqslant y, x : \alpha \to \beta, y : \beta, \Gamma \Rightarrow \Delta}{x \leqslant y, x : \alpha \to \beta, \Gamma \Rightarrow \Delta} \ (\to, l)$$

$$\frac{}{x : \bot, \Gamma \Rightarrow \Delta} \ (\bot, l) \qquad \frac{x \leqslant y, y : \alpha, \Gamma \Rightarrow \Delta, y : \beta}{\Gamma \Rightarrow \Delta, x : \alpha \to \beta} \ (\to, r)$$

$$\frac{x \leqslant x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \ (Ref) \qquad \frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}{x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta} \ (Trans)$$

The Baha'i method for labelled calculus

- hypersequent calculi
- labelled sequent calculi
  For intermediate logics
  - Base system: Labelled calculus *G3I*
  - Properties of intermediate logics: frame conditions (first-order formulas of classical logic)
  - Generated Rules: structural labelled rules
- display calculi

From frame conditions to labelled rules

Classification of frame conditions:

- Observation: Frame conditions are formulas of classical logic
- Polarities: $\exists$: positive – $\forall$: negative

Definition

- formulas in Σ_0, Π_0 are quantifier-free
- if ϕ is equivalent to $\exists\overline{x}\psi$ were $\psi \in \Pi_n$ then $\phi \in \Sigma_{n+1}$
- if ϕ is equivalent to $\forall\overline{x}\psi$ were $\psi \in \Sigma_n$ then $\phi \in \Pi_{n+1}$

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Theorem

Any formula within the class Π_2 (i.e. $\forall \overline{x} \exists \overline{y} \psi$) can be transformed into a set of equivalent structural rules that preserve cut-elimination when added to G3I (with additional axioms $x \leqslant y, \Gamma \Rightarrow \Delta, x \leqslant y$).

From frame conditions to labelled rules II

Example (Jankov logic)

Frame condition $(\forall xyz(x \leqslant y \wedge x \leqslant z) \rightarrow \exists w(y \leqslant w \wedge z \leqslant w))$

Invertibility $$\frac{}{\Rightarrow \forall xyz((x \leqslant y \wedge x \leqslant z) \rightarrow \exists w(y \leqslant w \wedge z \leqslant w))}$$

Ackermann Lemma $$\frac{}{x' \leqslant y', x' \leqslant z' \Rightarrow \exists w(y' \leqslant w \wedge z' \leqslant w)}$$

Invertibility $$\frac{\exists w(y' \leqslant w \wedge z' \leqslant w), \Gamma \Rightarrow \Delta}{x' \leqslant y', x' \leqslant z', \Gamma \Rightarrow \Delta}$$

Invertibility $$\frac{y' \leqslant w' \wedge z' \leqslant w', \Gamma \Rightarrow \Delta}{x' \leqslant y', x' \leqslant z', \Gamma \Rightarrow \Delta}$$

Equivalent rule $$\frac{y' \leqslant w', z' \leqslant w', \Gamma \Rightarrow \Delta}{x' \leqslant y', x' \leqslant z', \Gamma \Rightarrow \Delta}$$

Relationships with the state of the art

Frame conditions following the “geometric axiom scheme”

$$\forall \overline{x}(\neg P_1 \vee \cdots \vee \neg P_m \vee \exists \overline{y}_1(\bigwedge Q_1) \vee \cdots \vee \exists \overline{y}_n(\bigwedge Q_n))$$

can be converted into structural rules (S. Negri, 2004)

$$\frac{\overline{Q}_1[y_1/x_1], \overline{P}, \Gamma \Rightarrow \Delta \quad \cdots \quad \overline{Q}_k[y_k/x_k], \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta}$$

($\overline{Q}_i, \overline{P}$ are sets of atoms) that preserve cut-elimination

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- These formulas are $\in \Pi_2$
- Our rules for geometric formulas are the same

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($\overline{Q}_i, \overline{P}$ are sets of atoms) that preserve cut-elimination

- These formulas are $\in \Pi_2$
- Our rules for geometric formulas are the same

Are there frame conditions $\in \Pi_2$ that are not equivalent to any geometric formula?

The Bahai's method at work III

- hypersequent calculus
- labelled sequent calculus
- display calculus

(-, R. Ramanayake). *WOLLIC* 2013.

Display Calculus

Gentzen Sequent: $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$
$(A_1 \wedge \ldots \wedge A_n \Rightarrow B_1 \vee \ldots \vee B_m)$

Belnap's idea ('82) : look at $\Rightarrow$ as a deducibility relation between finite possible complex data (structures)

Display Calculus

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Definition (Display Sequent)

$X \Rightarrow Y$, where X, Y are structures which are built from formulae using structural connectives.

Display Calculus

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$(A_1 \wedge \ldots \wedge A_n \Rightarrow B_1 \vee \ldots \vee B_m)$

Belnap's idea ('82) : look at $\Rightarrow$ as a deducibility relation between finite possible complex data (structures)

Definition (Display Sequent)

$X \Rightarrow Y$, where X, Y are structures which are built from formulae using structural connectives.

Display property: given a display sequent $X \Rightarrow Y$ and any occurrence of a substructure Z in the sequent, that occurrence can be displayed as $Z \Rightarrow U$ or as $U \Rightarrow Z$ using structural rules.

Example: Display calculus δHB for Bi-Int

(Kamide and Wansing 2012)

- $\mathbf{I}$ is $\bot$
- $\bullet$ is "," on the RHS ($= \vee$) and $\rightarrow^d$ on the LHS
- $\circ$ is $\rightarrow$ on the RHS and $\wedge$ on the LHS

(Some of the) Logical rules:

$$\frac{\mathbf{I} \vdash X}{\top \vdash X}\ \top l \qquad \frac{X \vdash \mathbf{I}}{X \vdash \bot}\ \bot r$$

$$\frac{X \vdash A \qquad X \vdash B}{X \vdash A \wedge B}\ \wedge r \qquad \frac{A \vdash X \qquad B \vdash X}{A \vee B \vdash X}\ \vee l$$

$$\frac{X \vdash A \qquad Y \vdash B}{A \rightarrow B \vdash X \circ Y} \rightarrow l \qquad \frac{X \vdash A \circ B}{X \vdash A \rightarrow B} \rightarrow r \qquad \frac{B \bullet A \vdash X}{B \rightarrow_d A \vdash X} \rightarrow_d l$$

$$\frac{X \vdash B \qquad Y \vdash A}{X \bullet Y \vdash B \rightarrow_d A} \rightarrow_d r$$

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- $\circ$ is $\rightarrow$ on the RHS and $\wedge$ on the LHS

Display rules

$$\frac{\dfrac{Y \vdash X \circ Z}{X \circ Y \vdash Z}}{X \vdash Y \circ Z} \qquad \frac{\dfrac{X \bullet Y \vdash Z}{X \vdash Y \bullet Z}}{X \bullet Z \vdash Y} \qquad \frac{\dfrac{\mathbf{I} \circ X \vdash Y}{X \vdash Y}}{X \circ \mathbf{I} \vdash Y} \qquad \frac{\dfrac{X \vdash Y \bullet \mathbf{I}}{X \vdash Y}}{X \vdash \mathbf{I} \bullet Y}$$

Structural rules

$$\frac{X \vdash Y}{X \vdash Y \bullet Z} \qquad \frac{X \vdash Y}{X \circ Z \vdash Y} \qquad \frac{X \vdash Y \bullet Z}{X \vdash Z \bullet Y} \qquad \frac{X \circ Z \vdash Y}{Z \circ X \vdash Y}$$

$$\frac{X \vdash Y \bullet Y}{X \vdash Y} \qquad \frac{X \circ X \vdash Y}{X \vdash Y} \qquad \frac{X \vdash (Y \bullet Z) \bullet U}{X \vdash Y \bullet (Z \bullet U)} \qquad \frac{(X \circ Y) \circ Z \vdash U}{X \circ (Y \circ Z) \vdash U}$$

The Baha'i method for display calculus

- hypersequent calculi
- labelled sequent calculi
- display calculi
  For intermediate logics
  - Base system: the calculus δHB for (Bi)intuitionistic logic in (Kamide and Wansing 2012)
  - Properties of intermediate logics: Hilbert axioms
  - Generated Rules: structural display rules

The Baha'i method for display calculus

(Step 1): Classification of axioms based on the polarity of connectives

(Step 2): Transformation procedure

(Step 3): Completion of the generated rules

From Hilbert axioms to display rules

In the calculus δHB for intuitionistic logic all rules for connectives are invertible but $(\rightarrow, I)$

Classification:

$$\mathcal{I}_0 ::= \text{atomic formulae}$$

$$\mathcal{I}_{n+1} ::= \bot \mid \top \mid \mathcal{I}_n \rightarrow \mathcal{I}_{n+1} \mid \mathcal{I}_{n+1} \wedge \mathcal{I}_{n+1} \mid \mathcal{I}_{n+1} \vee \mathcal{I}_{n+1}$$

From Hilbert axioms to display rules

In the calculus δHB for intuitionistic logic all rules for connectives are invertible but $(\to, \mathit{l})$

Classification:

$$\mathcal{I}_0 ::= \text{atomic formulae}$$
$$\mathcal{I}_{n+1} ::= \bot \mid \top \mid \mathcal{I}_n \to \mathcal{I}_{n+1} \mid \mathcal{I}_{n+1} \wedge \mathcal{I}_{n+1} \mid \mathcal{I}_{n+1} \vee \mathcal{I}_{n+1}$$

Theorem

Each axiom within the class $\mathcal{I}_2$ can be transformed into equivalent structural display rules which preserve cut-elimination when added to δHB.

From Hilbert axioms to display rules

$\mathbf{I}$ is $\bot$, $\bullet$ is $\vee$ (RHS) and $\rightarrow^d$ (LHS), $\circ$ is $\rightarrow$ (RHS) and $\wedge$ (LHS)

Example (Jankov logic – Axiom $\sim\phi \vee \sim\sim\phi$)

Invertibility $\dfrac{}{\mathbf{I} \bullet (\sim\phi \circ \mathbf{I}) \vdash \phi \circ \mathbf{I}}$

Display Property $\dfrac{}{\phi \vdash (\mathbf{I} \bullet (\sim\phi \circ \mathbf{I})) \circ \mathbf{I}}$

Ackermann Lemma $\dfrac{X \vdash \phi}{X \vdash (\mathbf{I} \bullet (\sim\phi \circ \mathbf{I})) \circ \mathbf{I}}$

Display Property $\dfrac{X \vdash \phi}{\sim\phi \vdash (\mathbf{I} \bullet (X \circ \mathbf{I})) \circ \mathbf{I}}$

Ackermann Lemma $\dfrac{X \vdash \phi \qquad Y \vdash \sim\phi}{Y \vdash (\mathbf{I} \bullet (X \circ \mathbf{I})) \circ \mathbf{I}}$

Equivalent Rule $\dfrac{X \vdash Y \circ \mathbf{I}}{Y \vdash (\mathbf{I} \bullet (X \circ \mathbf{I})) \circ \mathbf{I}}$

Summary

A general method to introduce cut-free calculi, starting from

- Hilbert axioms (hypersequent and display calculi)
- Frame conditions (labelled calculi)

The method

- provides a unified treatment of very different formalisms
- provides (infinitely many) new cut-free systems
- subsumes many results proved for individual logics

(Some) Open problems & work in progress I

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- Conquering higher levels in the hierarchies.

By now:

class $\mathcal{P}_3$ for hypersequent calculi

$$
\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[r] \arrow[dr] & \mathcal{P}_2 \arrow[r] \arrow[dr] & \mathcal{P}_3 \arrow[r] \arrow[dr] & \mathcal{P}_4 \arrow[r] \arrow[dr] & \cdots \\
\mathcal{N}_0 \arrow[r] \arrow[ur] & \mathcal{N}_1 \arrow[r] \arrow[ur] & \mathcal{N}_2 \arrow[r] \arrow[ur] & \mathcal{N}_3 \arrow[r] \arrow[ur] & \mathcal{N}_4 \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}
$$

$\mathcal{I}_2$ for display calculi and $\forall\exists$ for labelled calculi

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\end{tikzcd}$$

$\mathcal{I}_2$ for display calculi and $\forall\exists$ for labelled calculi

Observation 1: All (axiomatizable) intermediate logics are within the classes $\mathcal{N}_3/\mathcal{I}_3$ (Zakharyaschev's canonical formulas)

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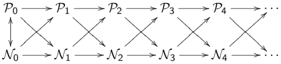

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Observation 2: Though $(\mathrm{Bd}_2) \notin \mathcal{P}_3$ we could find a **logical rule**

$$\frac{G \mid \Gamma', \Gamma \Rightarrow \Delta' \qquad G \mid \Gamma, A \Rightarrow B, \Delta}{G \mid \Gamma' \Rightarrow \Delta' \mid \Gamma \Rightarrow A \to B, \Delta}\ (bd_2)^*$$

(Some) Open problems & work in progress II

- First-order, modal logics, ...
- Extending the method to other (all?) general-purpose formalisms
- "Applications" of the generated calculi:
  E.g.
  - completeness of the formalized logics w.r.t. a semantics with truth-values in $[0,1]$ (hypersequent calculi)
  - decidability via suitable translations into nested sequents (display logic)
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