# How to Cover without Lifting Relations

Dirk Pattinson and Luigi Santocanale

Imperial College London
Universite de Provence

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# The Classical Cover Modality

## Standard Syntax of Modal Logic

$$\mathcal{L}_{\Box,\Diamond} \ni \phi, \psi ::= p \mid \phi \wedge \psi \mid \phi \vee \psi \mid \neg\phi \mid \Box\phi \mid \Diamond\phi \qquad (p \in V)$$

## Modal Logic in Terms of the Cover Modality

$$\mathcal{L}_{\nabla} \ni \phi, \psi ::= p \mid \phi \wedge \psi \mid \phi \vee \psi \mid \neg\phi \mid \nabla\mathbf{\Phi} \qquad (p \in V, \Phi \subseteq_f \mathcal{L}_{\nabla})$$

## Semantics

Suppose $\mathbb{M} = (W, \sigma : W \to \mathcal{P}(W), \pi : W \to \mathcal{P}(V))$ is a Kripke model.

$x \vDash \nabla\Phi$ iff

- $\forall \phi \in \Phi \exists y \in \sigma(x). y \vDash \phi$
- $\forall y \in \sigma(x) \exists \phi \in \Phi. y \vDash \phi$

"$\Phi$ and the successors of $x$ mutually cover one another"

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# Why the Cover Modality?

## Back and Forth Translation

**Forth.**

$$\nabla\Phi \equiv \Box \bigvee \phi \wedge \bigvee_{\phi\in\Phi} \Diamond\phi$$

**Back.**

$$\Box\phi \equiv \nabla\{\phi\} \vee \nabla\varnothing$$

$$\Diamond\phi \equiv \nabla\{\phi, \top\}$$

(we don't loose any expressiveness)

## Correspondence between Syntax and Semantics

- Kripke models $(W, \sigma, \pi)$ come with a *structure* $\sigma : W \to \mathcal{P}(W)$
- $\nabla$-formulas come with a *constructor* $\mathcal{P}_f(\mathcal{L}_\nabla) \to \mathcal{L}_\nabla$

(*finite* powersets give *finitary* languges)

## General Recipe?

$$\frac{\text{Semantics: Structures } \sigma : W \to TW}{\text{Syntax: Constructors } \nabla : T_f\mathcal{L} \to \mathcal{L}}$$

# Experiment: Probabilistic Frames

## Discrete Probability Distributions

$$\mathcal{D}X = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1, \sharp\{x \in X \mid \mu(x) \neq 0\} < \infty\}$$

## Probabilistic Kripke Models

$$\mathbb{M} = (W, \sigma : W \to \mathcal{D}(W), \pi : W \to \mathcal{P}(V))$$

(these are discrete Markov chains)

## Probabilistic Modal Logic

$$\mathcal{L}_\nabla \ni \phi, \psi ::= p \mid \phi \wedge \psi \mid \phi \vee \psi \mid \nabla \boldsymbol{\Phi} \qquad (p \in V, \Phi \in \mathcal{D}(\mathcal{L}_\nabla))$$

(probability distributions over formulae are formulae)

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# Satisfaction for Probabilstic Modal Logic

## Semantics and the Magic Square

Sse $\mathbb{M} = (W, \sigma, \pi)$ is a probabilstic model and $\nabla\mu \in \mathcal{L}_\nabla$, i.e. $\mu \in \mathcal{D}(\mathcal{L}_\nabla)$.

Then $w \models \nabla\mu$ iff we can fill the 'magic square'

| | $w_1$ | $w_2$ | $\cdots$ | $w_k$ | $\Sigma$ |
|---|---|---|---|---|---|
| $\phi_1$ | | | | | $q_1$ |
| $\vdots$ | | | | | $\vdots$ |
| $\phi_n$ | | | | | $q_n$ |
| $\Sigma$ | $p_1$ | $p_2$ | $\ldots$ | $p_k$ | |

- $p_j = \sigma(w)(x_j)$ is prob of $x_j$
- $q_i = \mu(\phi_i)$ is prob of $\phi_i$
- $w/\phi$-entry is $0$ if $x \not\models \phi$

can be filled according to the rules on the right.

## Question.

How far does this generalisation carry? Can we automagically construct magic squares?

# White Covers: The General Principle

## Definition ($T$-models)

Suppose $T : \mathsf{Set} \to \mathsf{Set}$ is a *functor*. Then $T$-models are triples $(W, \sigma, \pi)$ with $\sigma : W \to TW$ and $\pi : W \to \mathcal{P}(W)$.

## Definition ($T$-Language)

Write $T_f(X) = \bigcup\{TY \mid Y \subseteq_f X\}$ for the *finitary part* of $T$.

$$\mathcal{L}^T_\nabla \ni \phi, \psi ::= p \mid \phi \wedge \psi \mid \phi \vee \psi \mid \neg\phi \mid \nabla\boldsymbol{\Phi} \qquad (p \in V, \Phi \in T_f\mathcal{L}_\nabla)$$

## Definition (Semantics)

If $R \subseteq X \times Y$ is a relation, write

$$\hat{T}(R) = \{(s,t) \in TX \times TY \mid \exists w \in TR. T\pi_1(w) = s \text{ and } T\pi_2(w) = t\}$$

for the *relation lifting* of $T$ and put

$$w \models \nabla\Phi \iff (\sigma(w), \Phi) \in \hat{T}(\models)$$

# Results and Limitations

## Some Results (*terms and conditions apply*)

- Bisimulation invariance and Hennesy-Milner Property [Moss]
- Complete Axiomatisation(s) [Kupke, Kurz, Venema]
- Fixpoint Logics / Distributive Law [Venema]

## Limitations: Compatibility with Relational Composition

$$\text{Required:} \qquad \hat{T}(R \circ S) = \hat{T}R \circ \hat{T}S$$

where $\hat{T}$ is the relation lifting of $T$ (even for bisimulation invariance).

## Examples (*t's & c's fail*)

- Neighbourhood frames: $W \to \mathcal{P}\mathcal{P}(W)$
- Monotone nbhd frames: $W \to \{N \in \mathcal{P}\mathcal{P}(W) : N \text{ upclosed}\}$
- Selection function frames: $W \to (\mathcal{P}(W) \to \mathcal{P}(W))$

# Our Approach

**White Nablas: $\mathcal{L}_\nabla$**

**Syntax.**

$$\nabla : T_f \mathcal{L}_\nabla^T \to \mathcal{L}_\nabla$$

**Semantics.**

$$x \models \nabla\Phi \iff \sigma(x)\hat{T}(\models)\Phi$$

**Black Nablas: $\mathcal{L}_\blacktriangledown$**

**Syntax.**

$$\blacktriangledown : T\mathcal{C}(\Sigma) \to \mathcal{L}_\blacktriangledown^T$$

**Semantics.**

$$x \models \blacktriangledown\Phi \iff T(t) \circ \sigma(x) = \Phi$$

($\Sigma \subseteq_f \mathcal{L}_\blacktriangledown^T$, $\mathcal{C}(\Sigma)$ are $\neg$-complete subsets and t is the local theory map.)

## Conceptual Digression

- satisfaction for $\mathcal{L}_\nabla$ involves $\hat{T}(\models)$ which can fail
- satisfaction for $\mathcal{L}_\blacktriangledown$ involves $T(\mathrm{t})$ which always works

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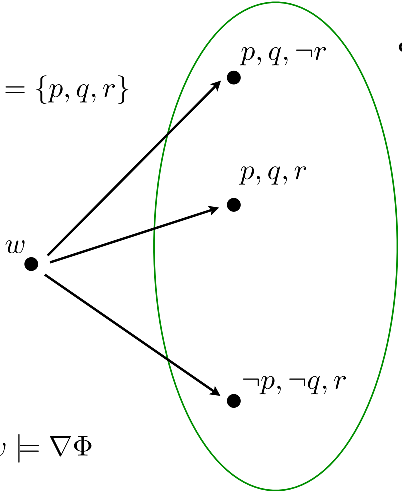
# Example 1: ▼ for Kripke Frames

$\Sigma = \{p, q, r\}$




$\Phi = \{ \ \{p, q, \neg r\}$

$\{p, q, r\}$

$\{\neg p, \neg q, r\} \}$

$w \models \nabla\Phi$

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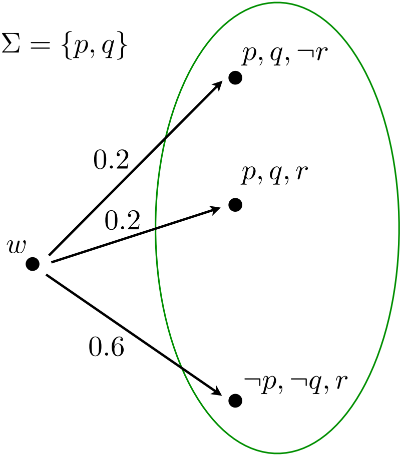
# Example 2: ▼ for Probabilistic Frames

$\Sigma = \{p, q\}$




$$\Phi = \quad 0.2 \cdot \{p, q\} + \quad 0.2 \cdot \{p, q\} + \quad 0.6 \cdot \{\neg p, \neg q\}$$

$$\begin{aligned} w \quad &\models \nabla 0.2 \cdot \{p, q\} + 0.2 \cdot \{p, q\} + 0.6 \cdot \{\neg p, \neg q\} \\ &= \nabla 0.4 \cdot \{p, q\} + 0.6 \cdot \{\neg p, \neg q\} \end{aligned}$$

# Black Nablas, Formally

**Definition (Syntax)**

Suppose $T : \mathsf{Set} \to \mathsf{Set}$.

$$\mathcal{L}^T_\blacktriangledown \ni \phi, \psi ::= p \mid \phi \wedge \psi \mid \phi \vee \psi \mid \neg\phi \mid \blacktriangledown\Phi$$

where $p \in V$ and $\Phi \in T\mathcal{C}(\Sigma)$ for $\Sigma \subseteq_f \mathcal{L}^T_\blacktriangledown \setminus \neg\mathcal{L}^T_\blacktriangledown$.

(recall $\mathcal{C}(\Sigma) = \{\Delta \subseteq \Sigma \cup \neg\Sigma \mid \forall\phi \in \Sigma.(\phi \in \Delta$ or $\neg\phi \in \Delta\})$)

**Definition (Semantics)**

Given a $T$-model $\mathbb{M} = (W, \sigma : W \to TW, \pi : W \to \mathcal{P}(V))$, put

$$w \vDash \blacktriangledown\Phi \iff T(\mathsf{t}\restriction_\Sigma) \circ \sigma(w) = \Phi$$

where $\mathsf{t}\restriction_\Sigma : w \mapsto \{\phi \in \Sigma \cup \neg\Sigma \mid \mathbb{M}, w \vDash \phi\}$ is the ($\Sigma$-) *local theory map*.

# Examples and Questions

## Examples

- Kripke frames
- *neighbourhood frames*
- *monotone nbhd frames*
- probabilistic frames
- *conditional frames*
- *etc.*

## What is the benefit of $\blacktriangledown$?

- Bisimulation invariance and the Hennessy-Milner property
- The finite model property
- Conjunction and Negation Elimination
- Simple Tableaux Calculus

in a *uniform* framework *not* requiring that relation lifting is well-behaved.

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# Translatability for Kripke Semantics

**Kripke Semantics: $\mathcal{L}_{\Box,\Diamond}$ to $\mathcal{L}^{\mathcal{P}}_{\blacktriangledown}$**

$$\Box\phi = \blacktriangledown\varnothing \vee \blacktriangledown\{\{\phi\}\}$$
$$\Diamond\phi = \blacktriangledown\{\{\phi\}\} \vee \blacktriangledown\{\{\phi\},\{\neg\phi\}\}$$

(for us, this direction would be enough)

**Kripke Semantics: $\mathcal{L}^{T}_{\blacktriangledown}$ to $\mathcal{L}_{\Box,\Diamond}$**

$$c \models \blacktriangledown\Phi \iff c \models \Box \bigvee_{\alpha\in\Phi}\left(\bigwedge\alpha \wedge \neg\bigvee(\Sigma\setminus\alpha)\right) \wedge \bigwedge_{\alpha\in\Phi}\Diamond\left(\bigwedge\alpha\right) \wedge \neg\bigvee(\Sigma\setminus\alpha)$$

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# Translatability for Monotone Neighbourhood Frames

## Monotone Neighbourhood Models

$$\mathbb{M} = (W, \sigma : W \to \mathcal{N}(W), \pi : W \to \mathcal{P}(V))$$

(recall $\mathcal{N}(W) = \{N \in \mathcal{P}\mathcal{P}(X) \mid N \text{ upclosed}\}$)

## Semantics

$$\mathbb{M}, w \vDash \Box\phi \iff \{w' \mid w' \vDash \phi\} \in \sigma(w)\}$$

('$\phi$ is a neighbourhood of $w$')

## Translation: $\mathcal{L}_{\Box,\Diamond}$ to $\mathcal{L}^{\mathcal{P}}_{\blacktriangledown}$

$$\Box\phi \equiv \blacktriangledown \uparrow \{\alpha_0\} \vee \blacktriangledown \uparrow \{\alpha_0, \alpha_1\}$$

where $\alpha_0 = \{\{\phi\}\}$, $\alpha_1 = \{\neg\phi\}$.

# Bisimulation Invariance

## Definition ($T$-morphisms)

A *$T$-morphism* $f : (W, \sigma, \pi) \to (W', \sigma', \pi')$ is a map $f : W \to W'$ such that

$$\begin{array}{ccc} W & \xrightarrow{f} & W' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma'} \\ TW & \xrightarrow[Tf]{} & TW' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} W & \xrightarrow{f} & W' \\ {\scriptstyle\pi}\searrow & & \swarrow{\scriptstyle\pi'} \\ & \mathcal{P}(V) & \end{array}$$

commute. A pair $(w, w') \in W \times W'$ is *behaviourally equivalent* if it can be identified by a pair of $T$-morphisms.

## Special Case: Kripke Frames, i.e. $T = \mathcal{P}$

- $\mathcal{P}$-morphisms are $p$-morphisms aka functional bisimulations
- behavioural equivalence is bisimilarity

# From Behavioural to Logical Equivalence

## Proposition (Morphisms preserve Semantics)

*Let* $f : (W, \sigma, \pi) \to (W', \sigma', \pi')$ *be a a* $T$*-morphism. Then, for all* $\phi \in \mathcal{L}^T_{\blacktriangledown}$*:*

$$w \vDash \phi \iff f(w) \vDash \phi.$$

*Proof.* For $\phi = \blacktriangledown\Psi$: $w$ and $f(w)$ inductively have the same local theories.

## Corollary (Behavioural Equivalence implies Logical Equivalence)

*Suppose that* $(w, w')$ *are behaviourally equivalent. Then, for all* $\phi \in \mathcal{L}^T_{\blacktriangledown}$*:*

$$w \vDash \phi \iff w' \vDash \phi.$$

*Proof.* $w \vDash \phi \iff f(w) \vDash \phi \iff g(w') \vDash \phi \iff w' \vDash \phi.$

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# From Logical to Behavioural Equivalence

## Proposition

*If $T$ is finitary ($\simeq$ finitely branching) and $\sim$ is logical equivalence, then*

$$(W, \sigma, \pi) \to (W/\sim, \sigma/\sim, \pi/\sim)$$

*is a well-defined $T$-morphism for all $T$-models $(W, \sigma, \pi)$.*

*Proof.* Find $W_0 \subseteq_f W$ with $\sigma(w), \sigma(w') \in TW_0$ and observe that $\sim\restriction_{W_0 \times W_0}$ can be characterised by finitely many formulae.

## Corollary

*If $T$ is finitary, then logical and behavioural equivalence coincide.*

*Proof.* $w \sim w' \iff [w']_\sim = [w]_\sim$.

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# Coherent Models and the Truth Lemma

## Definition (Coherent Models)

Let $\Delta \subseteq \mathcal{L}^T_{\blacktriangledown}$ be negation closed. Then $\mathbb{M} = (W, \sigma, \pi)$ is *coherent* over $\Delta$ if

- $W = \{\Theta \subseteq \Delta \mid \Theta \text{ maximally satisfiable}\}$
- $\blacktriangledown \Phi \in \Theta \iff T(\mathsf{t} \restriction \Phi) \circ \sigma(\Theta) = \Phi$
- $p \in \Theta \iff p \in \pi(\Theta)$

## Lemma (Truth Lemma)

*If* $\mathbb{M} = (W, \sigma, \pi)$ *is coherent over* $\Delta$*, then*

$$\Theta \models \phi \iff \phi \in \Theta$$

*for all* $\phi \in \Delta$*.*

*Proof.* Standard induction on the structure of formulae.

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# Existence Lemma and the Small Model Property

**Lemma (Existence Lemma)**

*If $\Delta$ is finite and negation closed, then coherent models exist.*

*Proof.* For every maximally satisfiable subset $\Theta \subseteq \Delta$ pick $w \models \Theta$ and define $\sigma(\Theta)$ by replacing points with local theories.

**Corollary (Small Model Property)**

*If $\phi \in \mathcal{L}^T_{\blacktriangledown}$ is satisfiable, then $\phi$ is satisfiable in an exponential-size model.*

*Proof.* Choose $\Delta$ to consist of the subformulas of $\phi$ and their negations.

# Conjunction Elimination

**Definition (Conjunction under $T$)**

Let $\Phi_i \in T\mathcal{C}(\Sigma_i)$ for $i = 1, 2$. Then

$$\Phi_1 \wedge \Phi_2 = \{\Delta \in T\mathcal{C}(\Sigma_1 \cup \Sigma_2) \mid T(\lambda\Theta.\Theta \cap (\Sigma_i \cup \neg\Sigma_i)) = \Phi_i, i = 1, 2\}$$

denote the 'conjunction under $T$' of $\Phi_1$ and $\Phi_2$.

*Note.* Conjunction under $T$ gives *all consistent possibilities* to satisfy both $\blacktriangledown\Phi_1$ and $\blacktriangledown\Phi_2$.

**Lemma (Conjunction Elimination Lemma)**

$$w \vDash \blacktriangledown\Phi_1 \wedge \blacktriangledown\Phi_2 \iff \exists\Psi \in \Phi_1 \wedge \Phi_2 . w \vDash \blacktriangledown\Psi$$

*Proof.* If $\blacktriangledown\Phi_1 \wedge \blacktriangledown\Phi_2$ is satisfiable, put $\Psi = T(\mathsf{t} \restriction_{\Sigma_1 \cup \Sigma_2} \circ \sigma(w))$.

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# Negation Elimination

## Definition (Negation under $T$)

Let $\Phi \in T\mathcal{P}(\Sigma)$. Then

$$\neg\Phi = \{\Delta \in T\mathcal{C}(\Sigma) \mid \Delta \neq \Phi\}$$

denotes the *negation under* $T$ of $\Phi$.

*Note.* Negation under $T$ gives *all possibilities* to satisfy $\neg \blacktriangledown \Phi$.

## Lemma (Negation Elimination)

$$w \models \neg \blacktriangledown \Phi \iff \exists \Psi \in \neg\Phi . w \models \Psi$$

*Proof.* Put $\Psi = T(\mathrm{t} \restriction_{\Sigma}) \circ \sigma(w)$.

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# Tableaux Calculus

## Extra Assumption

Suppose that $TX$ is finite whenever $X$ is finite.

## Prolegomena

*Sequents.* finite subsets $\Gamma, \Delta, \ldots$ of $\mathcal{L}^T_\blacktriangledown$, read conjunctively.

*Tableaux.* Sequent labelled trees construced according to rules (below)

*Closed Tableaux.* Maximal such trees.

*Completeness.* Unsatisfiability of $\Gamma \iff$ existence of closed tableau with root $\Gamma$.

## Tableau Rules: Propositional Part

$$(\wedge)\frac{\Gamma, \phi \wedge \psi}{\Gamma, \phi, \psi} \qquad (\vee)\frac{\Gamma, \phi \vee \psi}{\Gamma, \phi \quad \Gamma, \psi} (\mathsf{Ax})\frac{\Gamma, p, \neg p}{}$$

# Modal Rules

**Tableau Rules: Propositional Rules plus**

$$(\neg)\frac{\neg \blacktriangledown \Phi, \Gamma}{\{\blacktriangledown \Psi, \Gamma \mid \Psi \in \neg\Phi\}} \qquad (\wedge)\frac{\blacktriangledown \Phi_1 \wedge \blacktriangledown \Phi_2, \Gamma}{\{\blacktriangledown \Psi, \Gamma \mid \Psi \in \Phi_1 \wedge \Phi_2\}}$$

$$(\mathsf{Ing})\frac{\blacktriangledown \Phi, \Gamma}{\Psi}(\Gamma \subseteq V \cup \neg V \text{ consistent}, \Psi \in \mathsf{Ing}(\Phi))$$

where $\mathsf{Ing}(\Phi) = \bigcap\{\Psi \subseteq \mathcal{C}(\Sigma) \mid \Phi \in T\Psi$ are the *ingredients* of $\Phi$.

**Lemma (Invertibility)**

*The premise of a rule is satisfiable iff one if its conclusions is satisfiable.*

*Proof.* For (Ing), choose satisfying models for all conclusions and glue.

**Theorem (Completeness)**

*The tableau calculus for $\mathcal{L}^T_{\blacktriangledown}$ is complete.*

*Proof.* By invertibility it suffices to observe that all tableaux are finite.

# Conclusions

## Conceptual Achievements

- *uniform* framework for designing logics over large class of models
- 'standard' languages are encodable
- removed relation-lifting barrier

## Technical Achievements

- bisimulation invariance and Hennessy-Milber property
- small model property
- complete axiomatisation

## Loose Ends

- fixpoint extensions ($\leftarrow$ non-monotonicity?)
- extensions: nominals, global modality ...
- satisfiability games and automata