Decidability for Justification Logics Revisited

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joint work with
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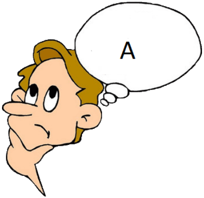
Modal Logic



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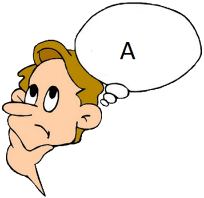

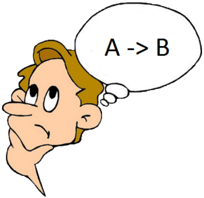

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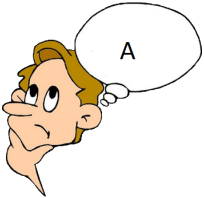

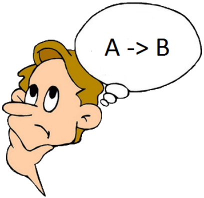

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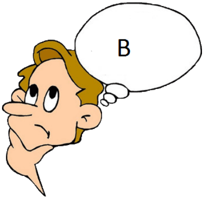

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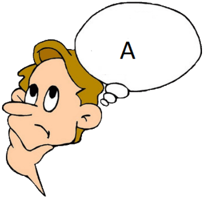

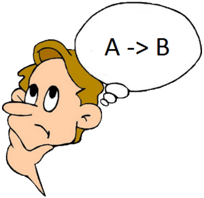

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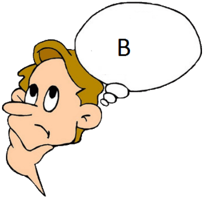
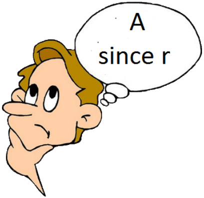
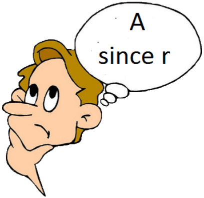


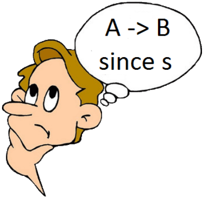
$$\Box A \quad \wedge \quad \Box(A \to B) \quad \to \quad \Box B$$

Justification Logic



Justification Logic





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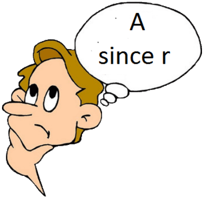

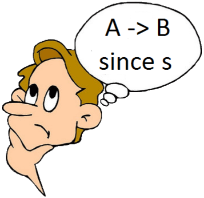

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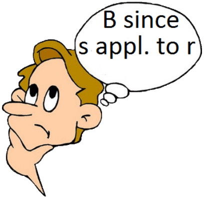

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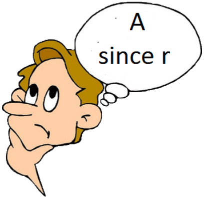

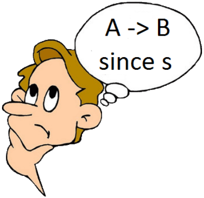

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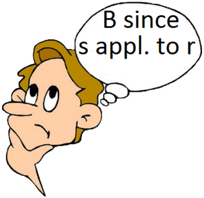

$$r : A \quad \wedge \quad s : (A \to B) \quad \to \quad s \cdot r : B$$

Syntax of Justification Logic

Logic

$JT4_{CS}$ is a justification counterpart of S4.

Justification terms

$$t ::= x \mid c \mid (t \cdot t) \mid (t + t) \mid !t$$

Formulas

$$A ::= p \mid \neg A \mid (A \to A) \mid t : A$$

Axioms for $\mathsf{JT4}_{\mathsf{CS}}$

- all propositional tautologies
- $t : (A \to C) \to (s : A \to t \cdot s : C)$ (application)
- $t : A \to t + s : A, \quad s : A \to t + s : A$ (sum)
- $t : A \to A$ (reflection)
- $t : A \to !t : t : A$ (introspection)

Deductive System

Constant specification

A constant specification CS is any subset

$$\mathsf{CS} \subseteq \{c : A \mid c \text{ is a constant and } A \text{ is an axiom}\}.$$

The deductive system $\mathsf{JT4}_{\mathsf{CS}}$ consists of the above axioms and the rules of modus ponens and axiom necessitation.

$$\frac{A \quad A \to B}{B} \qquad\qquad \frac{c : A \in \mathsf{CS}}{c : A}$$

Semantics

Definition (Admissible Evidence Relation)

Let CS be a constant specification. An admissible evidence relation $\mathcal{E}$ is a subset of $\mathsf{Tm} \times \mathsf{Fm}$ such that:

1. if $c : A \in CS$, then $(c, A) \in \mathcal{E}$
2. if $(s, A) \in \mathcal{E}$ or $(t, A) \in \mathcal{E}$, then $(s + t, A) \in \mathcal{E}$
3. if $(s, A \to B) \in \mathcal{E}$ and $(t, A) \in \mathcal{E}$, then $(s \cdot t, B) \in \mathcal{E}$
4. if $(t, A) \in \mathcal{E}$, then $(!t, t : A) \in \mathcal{E}$

Definition (Model)

Let CS be a constant specification. A *model* is a pair $\mathcal{M} = (\mathcal{E}, \nu)$ where

- $\mathcal{E}$ is an admissible evidence relation,
- $\nu \subseteq \mathsf{Prop}$ is a valuation.

Soundness and Completeness

Definition (Satisfaction relation)

Let $\mathcal{M} = (\mathcal{E}, \nu)$ be a model.

1. $\mathcal{M} \Vdash F$ is defined as usual for propositions and boolean connectives
2. $\mathcal{M} \Vdash t : A$ if and only if
   1. $(t, A) \in \mathcal{E}$ and
   2. $\mathcal{M} \Vdash A$

Theorem

Let CS *be a constant specification. A formula A is derivable in* $\mathsf{JT4}_{\mathsf{CS}}$ *if and only if A is valid.*

Decidability

Lemma

Let a finitely axiomatizable logic L *be sound and complete with respect to a class of models* $\mathcal{C}$*, such that*

1. *the class* $\mathcal{C}$ *is recursively enumerable, and*
2. *the binary relation* $\mathcal{M} \Vdash F$ *between formulae and models from* $\mathcal{C}$ *is decidable.*

Then L *is decidable.*

Finitely Generated Models

Definition

1. An *evidence base* $\mathcal{B}$ is a subset of $\mathsf{Tm} \times \mathsf{Fm}$.
2. $\mathcal{E}_{\mathcal{B}}$ is the least admissible evidence relation containing $\mathcal{B}$.

Definition (Finitely generated model)

Let CS be a finite constant specification. Let $\mathcal{B}$ be a finite evidence base and ν be a finite valuation. Then we call $\mathcal{M}_{\mathcal{B}} = (\mathcal{E}_{\mathcal{B}}, \nu)$ a *finitely generated model*.

Theorem

1. *The satisfaction relation for finitely generated models is decidable.*
2. *The class of finitely generated models is recursively enumerable.*

Φ-Generated Submodel

Definition

Let $\mathcal{M} = (\mathcal{E}, \nu)$ be a model and Φ some set of formulae closed under subformulae. The Φ-generated submodel $\mathcal{M} \restriction \Phi$ of $\mathcal{M}$ is defined by $(\mathcal{E} \restriction \Phi, \nu \restriction \Phi)$ where

1. $\mathcal{E} \restriction \Phi$ is the evidence relation generated from the base $\mathcal{B}_\Phi$ given by $(t, F) \in \mathcal{B}_\Phi$ iff $t : F \in \Phi$ and $(t, F) \in \mathcal{E}$,
2. $\nu \restriction \Phi$ is given by $p_i \in \nu \restriction \Phi$ iff $p_i \in \Phi$ and $p_i \in \nu$.

Lemma

Let $\mathcal{M} = (\mathcal{E}, \nu)$ be a model and Φ be a set of formulae closed under subformulae. Let $\mathcal{M} \restriction \Phi$ be the Φ-generated submodel of $\mathcal{M}$. Then for all formulae $F \in \Phi$ we have

$$\mathcal{M} \restriction \Phi \Vdash F \text{ if and only if } \mathcal{M} \Vdash F.$$

Decidability

Theorem

Let CS *be a finite constant specification.* $\mathsf{JT4}_{\mathsf{CS}}$ *is complete with respect to finitely generated models.*

Corollary

$\mathsf{JT4}_{\mathsf{CS}}$ *is decidable for finite constant specifications* CS.

Problem: Infinite Constant Specifications

Be careful

Kuznets: There is a decidable CS such that $\mathsf{JT4}_{\mathsf{CS}}$ is undecidable.

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Theorem

$\mathsf{JT4}_{\mathsf{CS}}$ *is decidable for schematic constant specifications* CS.

Admissible evidence relation stores formula schemes.

Use unification in the case of application.

Then $\mathcal{E}$ is decidable.

Problem: D-axiom

D-Axiom: $\neg t : \bot$ for all terms t

Semantically: $(t, \bot) \notin \mathcal{E}$

Question: How to enumerate models?

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Semantically: $(t, \bot) \notin \mathcal{E}$

Question: How to enumerate models?

Use F-models, which combine traditional Kripke-frames with evidence relation.

There D-axiom corresponds to frame condition and not to a condition on $\mathcal{E}$

Use filtrations to get finitary F-models

Theorem

$\mathsf{JD4}_{\mathsf{CS}}$ *is decidable for schematic and axiomatically appropriate constant specifications* CS.

Problem: Negative Introspection

5-axiom: $\neg t : A \to ?t : \neg t : A$

Semantically: if $(t, A) \notin \mathcal{E}$, then $(?t, \neg t : A) \in \mathcal{E}$

$\mathsf{JT45}_{\mathsf{CS}}$ only sound wrt. strong models: $(t, A) \in \mathcal{E} \Longrightarrow \mathcal{M} \Vdash t : A$

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Need non-monotone inductive definition to generate models

Show that $\mathcal{E}$ and satisfaction relation are decidable

Show that it is decidable whether finitely generated model is strong

Thus finitely generated strong models are recursively enumerable

Show that submodel construction preserves strong models

Theorem

$\mathsf{JT45}_{\mathsf{CS}}$ *is decidable for finite constant specifications* CS.

Thank you!