2D products of modal logics: new results on axiomatisation and decision problems

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Bimodal logics

- **bimodal formulas:** $\varphi \; = \; p \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \Diamond_1\varphi \mid \Diamond_2\varphi$
- **Kripke frames:** $\mathfrak{F} = \langle W, R_1, R_2 \rangle$
- **(normal) bimodal logic:** a set of bimodal formulas
  - containing the axioms $\mathbf{K}_i$ $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
  - closed under the rules of **Substitution**, **Modus Ponens**, and **Necessitation**$_i$ $\varphi \,/\, \Box_i\varphi$ for $i = 1, 2$.

Example: $\textbf{Logic\_of}\ \mathcal{C} \; = \; \{\varphi \mid \forall\mathfrak{F} \in \mathcal{C},\ \mathfrak{F} \models \varphi\}$ for any class $\mathcal{C}$ of frames.

- A recursive set Σ of bimodal formulas **axiomatises** a bimodal logic L, if L is the smallest logic containing Σ.

Some well-known (uni)modal logics

$$\begin{aligned}
\mathbf{K} &= \textbf{Logic_of}\ \{\text{all frames}\}\\
\mathbf{K4} &= \textbf{Logic_of}\ \{\text{transitive frames}\}\\
\mathbf{S4} &= \textbf{Logic_of}\ \{\text{reflexive and transitive frames}\}\\
\mathbf{S5} &= \textbf{Logic_of}\ \{\text{equivalence frames}\}\\
\mathbf{K4.3} &= \textbf{Logic_of}\ \{\text{linear frames}\}\\
\mathbf{GL} &= \textbf{Logic_of}\ \{\text{transitive, irreflexive and Noetherian frames}\}\\
\mathbf{GL.3} &= \textbf{Logic_of}\ \{\text{linear, irreflexive and Noetherian frames}\}\\
\mathbf{Alt} &= \textbf{Logic_of}\ \{\text{functional frames}\}
\end{aligned}$$

Each of these logics is:

- **finitely axiomatisable**
- has the finite model propery (fmp), and **decidable**
  - polynomial fmp and **coNP-complete** : $\mathbf{S5}, \mathbf{K4.3}, \mathbf{GL.3}, \mathbf{Alt}$
  - exponential fmp and **PSPACE-complete** : $\mathbf{K}, \mathbf{K4}, \mathbf{S4}, \mathbf{GL}$

Product frames and products of modal logics

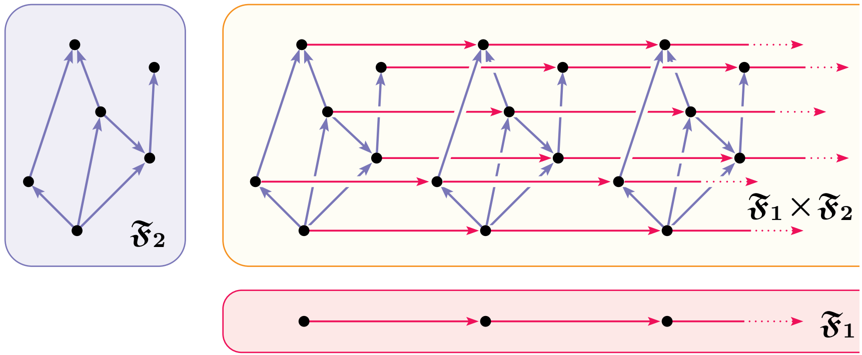
Segerberg 1973, Shehtman 1978, Gabbay-Shehtman 1998

For $\mathfrak{F}_1 = \langle W_1, R_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2 \rangle$,

the **product frame** is $\mathfrak{F}_1 \times \mathfrak{F}_2 = \langle W_1 \times W_2, R_h, R_v \rangle$, where

$\langle x, y \rangle \, R_h \, \langle x', y' \rangle$ iff
$x R_1 x'$ and $y = y'$

$\langle x, y \rangle \, R_v \, \langle x', y' \rangle$ iff
$y R_2 y'$ and $x = x'$



The **product** of two Kripke complete unimodal logics L_1, L_2 is the **bimodal logic**

$$L_1 \times L_2 \quad = \quad \textbf{Logic\_of}\ \{\mathfrak{F}_1 \times \mathfrak{F}_2 \mid \mathfrak{F}_1 \in \textbf{Fr}\ L_1,\ \mathfrak{F}_2 \in \textbf{Fr}\ L_2\}$$

Connections with other formalisms

- spatio-temporal logics
- dynamic topological logics
- temporal-epistemic logics, multi-agent systems
- modal and temporal description logics
- two-variable fragment of classical predicate logic
- representable cylindric algebras of dimension 2
- one-variable fragment of modal and intuitionistic predicate logics

Products vs. two-variable predicate logic

bimodal formula φ $\rightsquigarrow$ f-o formula φ^* with ≤ 2 variables

$$
\begin{array}{rcl}
p & \mapsto & P(x,y) \\
\varphi \vee \psi & \mapsto & \varphi^* \vee \psi^* \\
\neg\varphi & \mapsto & \neg\varphi^* \\
\Diamond_1\varphi & \mapsto & \exists x\ \varphi^* \\
\Box_1\varphi & \mapsto & \forall x\ \varphi^* \\
\Diamond_2\varphi & \mapsto & \exists y\ \varphi^* \\
\Box_2\varphi & \mapsto & \forall y\ \varphi^*
\end{array}
$$

φ is $\mathrm{S5} \times \mathrm{S5}$-satisfiable $\iff$ φ^* is f-o satisfiable

Modal algebras for $\mathbf{S5} \times \mathbf{S5}$:

representable diagonal-free cylindric algebras of dimension 2

General properties of $L = L_1 \times L_2$

- $\boldsymbol{L_1 \times L_2}$ is determined by products of **rooted** $\boldsymbol{L_i}$-frames.
- $\boldsymbol{L_1 \times L_2}$ is determined by its **countable** product frames, whenever the class of all $\boldsymbol{L_i}$-frames is elementary, for $\boldsymbol{i = 1, 2}$. $\quad (\mathbf{K}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}, \mathbf{K4.3}, \mathbf{Alt}, \dots)$
- $\boldsymbol{L_1 \times L_2}$ is **recursively enumerable**, whenever the class of all $\boldsymbol{L_i}$-frames has a recursive first-order axiomatisation, for $\boldsymbol{i = 1, 2}$. $\quad (\mathbf{K}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}, \mathbf{K4.3}, \mathbf{Alt}, \dots)$

  Not always: *Gabelaia-K-Wolter-Zakharyaschev 2005*

  $\mathbf{K4} \times \mathbf{GL.3} \qquad \mathbf{K4} \times$ **Logic_of** $\{\langle \omega, < \rangle\}$ — Π^1_1-**hard**

  Open: $\mathbf{K4} \times \mathbf{GL}\,?$ — **undecidable**

- $\boldsymbol{L_1 \times L_2}$ is **canonical**, whenever the class of all $\boldsymbol{L_i}$-frames is elementary, for $\boldsymbol{i = 1, 2}$. $\quad (\mathbf{K}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}, \mathbf{K4.3}, \mathbf{Alt}, \dots)$

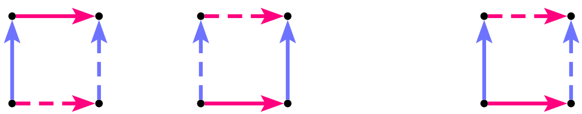
Axiomatising product logics: first steps

Given Kripke complete unimodal logics L_1 and L_2, their **commutator**

$$[L_1, L_2]$$

is the smallest bimodal logic containing L_i (for $\Diamond_i$) and the (Sahlqvist) formulas

$$\Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p \quad \text{and} \quad \Diamond_1\Box_2 p \rightarrow \Box_2\Diamond_1 p$$



commutativity and **confluence**: f-o ∀∃-conditions

- $[L_1, L_2] \subseteq L_1 \times L_2$
- The commutator of elementary logics is **canonical** (so **Kripke complete**).

  Not always even Kripke complete: *Gabelaia-K-Wolter-Zakharyaschev 2005*

  $[\mathbf{K4}, \mathbf{GL.3}]$ is incomplete

$$[L_1, L_2] \stackrel{?}{=} L_1 \times L_2$$ — **product matching?**

Product matching pairs

Gabbay–Shehtman 1998

Horn formula: $\forall x y \bar{z}\,(\Phi(x,y,\bar{z}) \to R(x,y))$ where Φ is positive

Horn axiomatisable modal logic: axiomatisable by modal formulas having Horn first-order correspondents $\mathbf{K},\ \mathbf{K4},\ \mathbf{S4},\ \mathbf{S5},\ \dots$

- If L_1 and L_2 are Horn axiomatisable then $[L_1, L_2] = L_1 \times L_2$
- $[\mathbf{Alt}, \mathbf{Alt}] = \mathbf{Alt} \times \mathbf{Alt}$

Universal but not Horn: **weak connectedness** (**linearity** in rooted frames)

$$\forall x y z\,(R(x,y) \land R(x,z) \to y = z \lor R(y,z) \lor R(z,y))$$

E.g., is $\mathrm{K4.3} \times \mathrm{K}$ product matching? Is it finitely axiomatisable?

Non-finitely axiomatisable 2D product logics

K–Marcelino 2010

- Let L be a bimodal logic such that
  - $\boxed{\mathbf{K4.3} \times \mathbf{K} \ \subseteq \ L}$ and
  - the product of $\langle \omega, < \rangle$ and an ω-fan is a frame for L.

  Then **every axiomatisation** of L must contain **infinitely many propositional variables.**

- Let L be a bimodal logic such that

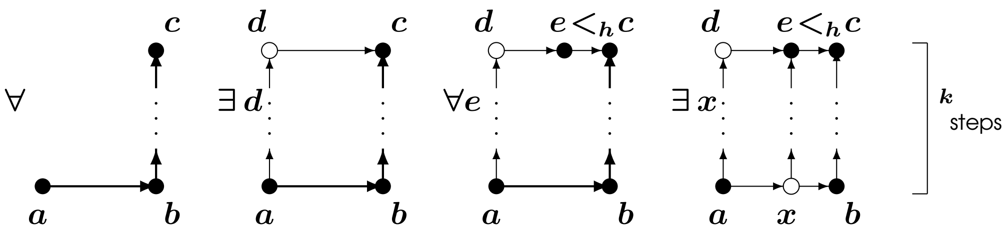
  $$\boxed{\mathbf{K4.3} \times \mathbf{K} \ \subseteq \ L \ \subseteq \ \textbf{Logic\_of} \ \{\langle \omega, < \rangle\} \times \mathbf{K}.}$$

  Then **every axiomatisation** of L must contain formulas of **arbitrarily large $\Diamond_2$-depth.**

- Same hold for reflexive versions.

What kind of axioms we need?

- Example of **arbitrarily large $\Diamond_2$-depth:** modally definable $\forall\exists\forall\exists$-conditions



- **Open:** Are $\mathbf{K4.3} \times \mathbf{K4.3}$ or $\mathbf{K4.3} \times \mathbf{S5}$ finitely axiomatisable?

  — known to be NOT product matching

- Logics like $\mathbf{K4.3} \times \mathbf{K}$, $\mathbf{K4.3} \times \mathbf{K4}$ are **non-finitely axiomatisable,** but **r.e.** and **canonical.**

  Open: More info on **how complex** an infinite axiomatisation could be.

  - Does the product logic have a **canonical axiomatisation**?
  - Is the class of **all** frames for the product logic **closed under ultraproducts**?
  - Can we use some modifications of successful **≥ 3D-techniques** here?

Finite model properties

Product logics are determined by classes of product frames, but there are **non-product frames** for product logics **!**

- A product logic $\boldsymbol{L_1} \times \boldsymbol{L_2}$ has the **product fmp** if any $\varphi \notin \boldsymbol{L_1} \times \boldsymbol{L_2}$ fails in a finite **product** frame for $\boldsymbol{L_1} \times \boldsymbol{L_2}$.
- A product logic $\boldsymbol{L_1} \times \boldsymbol{L_2}$ has the **(abstract) fmp** if any $\varphi \notin \boldsymbol{L_1} \times \boldsymbol{L_2}$ fails in a finite (not necessarily product) frame for $\boldsymbol{L_1} \times \boldsymbol{L_2}$.

$$\text{product fmp} \quad \Longrightarrow \quad \text{fmp}$$
$$\not\Longleftarrow$$

Known results on the decision problem and fmp

Gabbay–Shehtman 1998,2000, Gabbay-K-Wolter-Zakharyaschev 2004, Reynolds-Zakharyaschev 2001, Gabelaia-K-Wolter-Zakharyaschev 2005, ...

- Products with $\mathbf{S5}$**-like logics** are usually **decidable** and enjoy the **fmp** (typical complexity coNExpTime)

- Products with $\mathbf{K}$**-like logics** are usually **decidable** and enjoy the **fmp**

  Open: Is there an elementary decision algorithm for $\mathbf{K} \times \mathbf{K}$?

- Products with both components having **only transitive frames** are usually **undecidable** and have **no fmp**

  - If both component logics are determined by recursively first-order axiomatisable classes of frames then the product is **recursively enumerable** (like $\mathbf{K4} \times \mathbf{K4}$, $\mathbf{S4} \times \mathbf{K4.3}$)

  - But it can be even Π^1_1**-complete** (like $\mathbf{K4} \times \textbf{Logic\_of } \{\langle \omega, < \rangle\}$, $\mathbf{S4} \times \mathbf{GL.3}$)

Connecting the dimensions: products with diagonal constant

- We add a constant δ to the bimodal language with $\Diamond_1, \Diamond_2$
- Given frames $\mathfrak{F}_1 = \langle W_1, R_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2 \rangle$,

  their **δ-product** is $\mathfrak{F}_1 \times^\delta \mathfrak{F}_2 = \langle W_1 \times W_2, R_h, R_v, D \rangle$, where

  - $\langle W_1 \times W_2, R_h, R_v \rangle = \mathfrak{F}_1 \times \mathfrak{F}_2$
  - $D = \{\langle u, u \rangle \mid u \in W_1 \cap W_2\}$
- The **δ-product** of two Kripke complete unimodal logics L_1, L_2 is

  the **3-modal logic**

$$L_1 \times^\delta L_2 = \textbf{Logic\_of}\ \{\mathfrak{F}_1 \times^\delta \mathfrak{F}_2 \mid \mathfrak{F}_1 \in \textbf{Fr}\ L_1,\ \mathfrak{F}_2 \in \textbf{Fr}\ L_2\}$$

↑

in the language with $\Diamond_1, \Diamond_2, \delta$

What can the diagonal constant mean?

- **equality** in two-variable classical predicate logic

  (constant d_{01} in two-dimensional representable cylindric algebras)

- when reasoning about domains changing in time

  (under actions, belief change, etc.):

  δ can collect a set of special **time-stamped objects** such that

  - no special object is chosen twice, and
  - at every moment of time, at most one special object is chosen.

- . . . ?

Non-finitely axiomatisable 2D products with diagonal

Kikot 2010

Let L_i be any logic such that $\mathbf{K} \subseteq L \subseteq \mathbf{GL}$

Then **every** aximatisation of $L_1 \times^\delta L_2$ must contain **infinitely many propositional variables.**

Examples: $\mathbf{K} \times^\delta \mathbf{K},\ \mathbf{K} \times^\delta \mathbf{K4},\ \mathbf{K4} \times^\delta \mathbf{K4},\ \mathbf{GL} \times^\delta \mathbf{GL},\ \mathbf{K} \times^\delta \mathbf{GL}$

product matching without diagonal

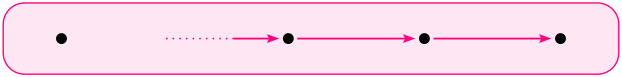
Undecidable 2D products with diagonal

K-Kikot 2011

Let $\mathfrak{F} = \langle \omega + 1, R \rangle$ where

$$R = \{\langle \omega, n \rangle \mid \text{ for all } n < \omega\} \cup \{\langle n+1, n \rangle \mid \text{ for all } n < \omega\}.$$

$\mathfrak{F}$:



$\mathfrak{F}^{tr}$: transitive closure of $\mathfrak{F}$

Let $\mathcal{C}$ be a class of δ-product frames such that $\mathfrak{F} \times^{\delta} \mathfrak{F}$ or $\mathfrak{F} \times^{\delta} \mathfrak{F}^{tr}$ is in $\mathcal{C}$.
Then **Logic_of** $(\mathcal{C})$ is **undecidable.**

Most surprising examples: $\mathbf{K} \times^{\delta} \mathbf{K}, \quad \mathbf{K} \times^{\delta} \mathbf{K4}$

2D products with diagonal: no fmp

K 2009

Let

- $\mathfrak{G} = \langle \omega + 1, S \rangle$ with $S = \{\langle \omega, n \rangle \mid n < \omega\}$ (ω-fan)
- $S^{\textit{refl}}$ = reflexive closure of S
- $S^{\textit{univ}}$ = the universal relation on $\omega + 1$

If $\mathcal{C}$ is any class of δ-product frames such that

- either $\mathfrak{F} \times^\delta \mathfrak{G} \in \mathcal{C}$
- or $\mathfrak{F} \times^\delta \mathfrak{G}^{\textit{refl}} \in \mathcal{C}$
- or $\mathfrak{F} \times^\delta \mathfrak{G}^{\textit{univ}} \in \mathcal{C}$

then **Logic_of** $(\mathcal{C})$ **does not** have the (abstract) fmp

Examples: $\mathbf{K} \times^\delta \mathbf{K}$, $\mathbf{K} \times^\delta \mathbf{K4}$, $\mathbf{K} \times^\delta \mathbf{S4}$, $\mathbf{K} \times^\delta \mathbf{S5}$

Why are these results surprising?

- Both $\mathbf{S5} \times \mathbf{S5}$ and $\mathbf{S5} \times^{\delta} \mathbf{S5}$ are **decidable** (coNExpTime-complete) and have the **product fmp**
- $\mathbf{K} \times \mathbf{S5}$ is also decidable, coNExpTime-complete, and has the **product fmp**
- $\mathbf{K} \times \mathbf{K}$ is also decidable and has the **product fmp**
- All known **undecidable** product-like logics have some kind of 'forward going' **universal modality**:

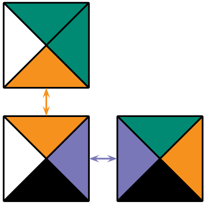
$\mathbf{K4} \times \mathbf{K4}$, $\mathbf{K} \times \mathbf{K}$ with universal modality

Undecidability proof

By reduction of the $\mathbb{N} \times \mathbb{N}$ tiling problem:

Given a finite set T of tile types $t = \langle \mathit{left}(t), \mathit{right}(t), \mathit{up}(t), \mathit{down}(t) \rangle$

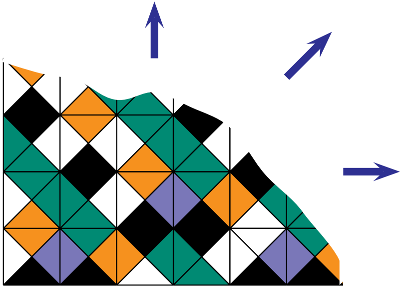
decide whether there exists $\tau : \mathbb{N} \times \mathbb{N} \to T$ such that, for all $i, j \in \mathbb{N}$,



$$\mathit{up}(\tau(i,j)) = \mathit{down}(\tau(i,j+1))$$

and

$$\mathit{left}(\tau(i,j)) = \mathit{right}(\tau(i+1,j)).$$

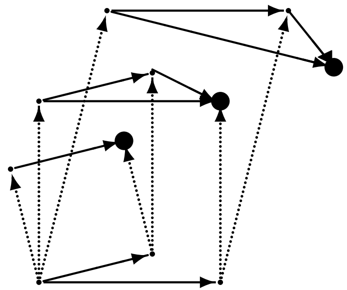


(Berger 1966): The $\mathbb{N} \times \mathbb{N}$ tiling problem is **undecidable**

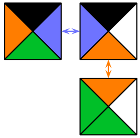
Undecidability proof: the formulas for $\mathrm{K} \times^\delta \mathrm{K}$

(γ): **Generating a $\mathbb{N} \times \mathbb{N}$-like grid 'upside down' so that all points are $\Box_1\Box_2$-accessible from the root (like $\mathfrak{F} \times \mathfrak{F}$):**

$$\Diamond_2\Diamond_1\delta$$
$$\Box_2\Diamond_1\Diamond_1\delta$$
$$\Box_1\Diamond_2\delta$$
$$\Box_1\Box_2(\delta \wedge \Diamond_1\top \longrightarrow \Diamond_2\top)$$
$$\Box_1\Box_2(\delta \longrightarrow \Box_1\Box_2\delta)$$



(ϑ): **encoding tiling rules**



$$\Box_1\Box_2 \bigvee_{t \in T} \Big(t \wedge \bigwedge_{t' \neq t} \neg t'\Big)$$
$$\Box_1\Box_2 \bigwedge_{\mathit{right}(t') \neq \mathit{left}(t)} \big(t \longrightarrow \Box_1\neg t'\big)$$
$$\Box_1\Box_2 \bigwedge_{\mathit{up}(t') \neq \mathit{down}(t)} \big(t \longrightarrow \Box_2\neg t'\big)$$

Claim. $(\vartheta \wedge \gamma)$ is **satisfied** in a δ-product frame in $\mathcal{C}$ iff T **tiles** $\mathbb{N}\times\mathbb{N}$

Future work on 2D products with diagonal

- Widen the scope of the undecidability theorem to those logics where
  the 'no fmp' theorem applies

Open: Is $\mathbf{K} \times^{\delta} \mathbf{S5}$ decidable?

- Explore possible connections with **3-dimensional product logics**

Open: Is it decidable whether a finite frame is a frame for $\mathbf{K} \times^{\delta} \mathbf{K}$?

- Explore possible connections with **relation algebras**

- Explore connections with other **undecidable extensions of products**
  say, with the universal modality ($=$ global consequence relation)

Connecting the dimensions: Segerberg operators

Segerberg 1973

- We add 3 unary modal operators $\sigma_0, \sigma_1, \sigma_2$ to our bimodal language
- Given frames $\mathfrak{F}_1 = \langle W_1, R_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2 \rangle$, their

  S-product is $\mathfrak{F}_1 \times^S \mathfrak{F}_2 = \langle W_1 \times W_2, R_h, R_v, S_0, S_1, S_2 \rangle$, where

  - $\langle W_1 \times W_2, R_h, R_v \rangle = \mathfrak{F}_1 \times \mathfrak{F}_2$
  - $$S_0 = \{\langle\langle u, v\rangle, \langle v, u\rangle\rangle \mid u, v \in W_1 \cap W_2\}$$
    $$S_1 = \{\langle\langle u, v\rangle, \langle v, v\rangle\rangle \mid u \in W_1,\ v \in W_1 \cap W_2\}$$
    $$S_2 = \{\langle\langle u, v\rangle, \langle u, u\rangle\rangle \mid u \in W_1 \cap W_2,\ v \in W_2\}$$

- (permutation and substitutions in 2-variable predicate logic)
- The **S-product** of two Kripke complete unimodal logics L_1, L_2 is

  the **5-modal logic**

$$L_1 \times^S L_2 = \textbf{Logic\_of}\ \{\mathfrak{F}_1 \times^S \mathfrak{F}_2 \mid \mathfrak{F}_1 \in \textbf{Fr}\ L_1,\ \mathfrak{F}_2 \in \textbf{Fr}\ L_2\}$$

↑

in the language with $\Diamond_1$, $\Diamond_2$, σ_0, σ_1, σ_2

Decidable and finitely axiomatisable S-products

Shehtman 2011

- If $\boldsymbol{L_1}$ and $\boldsymbol{L_2}$ are finitely **Horn** axiomatisable, then $\boldsymbol{L_1} \times^{\boldsymbol{S}} \boldsymbol{L_2}$ is **finitely axiomatisable.**
- $\mathbf{K} \times^{\boldsymbol{S}} \mathbf{K}$ has the **effective fmp,** and so **decidable.**

Some references

- D.M.Gabbay, A.Kurucz, F.Wolter, M.Zakharyaschev, Many-dimensional modal logics: theory and applications, Elsevier, 2003.
- D.M.Gabbay, V.Shehtman, *Products of modal logics, part I*, **J.IGPL**, 6:73–146, 1998
- D.Gabelaia, A.Kurucz, F.Wolter, M.Zakharyaschev, *Products of 'transitive' modal logics*, **J.Symb.Log.**, 70:993–1021, 2005
- A.Kurucz, Products of modal logics with diagonal constant lacking the finite model property, in: FroCoS 2009, LNAI vol.5749, Springer (2009), 295–302.
- S.Kikot, *Axiomatization of modal logic squares with distinguished diagonal*, **Math.Notes**, 88:238–250, 2010
- A.Kurucz, S.Marcelino, *Non-finitely axiomatisable two-dimensional modal logics*, (submitted), `http://www.dcs.kcl.ac.uk/staff/kuag/publi/linax.pdf`
- K.Segerberg, *Two-dimensional modal logic*, **J.Phil.Log.**, 2:77–96, 1973
- V.Shehtman, *Two-dimensional modal logics*, Math.Not.USSR Acad.Sci., 23:417–424, 1978
- V.Shehtman, *Squares of modal logics with additional connectives*, **Russian Math. Uspekhy**, 2011, No.5.