A few selected topics

Non tutto il male vien per nuocere.[1]

Italian saying.

[1] *Added in Proof.* "Not all bad things come to hurt you". Refers to an embarrassing mistake I made in the preceding talk, concerning the example with the tossing of a coin. In the present talk I took the opportunity to clarify the notions of satisfiability/consistency in Łukasiewicz logic which, as I realised also thanks to my own mistakes in the preceding talk, I had not explained clearly enough.

Clarifications on satisfiability and consistency in Ł

Notion	Definition	Description
α is satisfiable	$\exists w$ such that $w(\alpha) = 1$	α is 1-satisfiabile
α is consistent	$\exists \beta$ such that $\alpha \nvdash_{Ł} \beta$	α does not prove smthg.
α is unsatisfiable	$\forall w$ we have $w(\alpha) < 1$	α is not 1-satisfiable
α is inconsistent	$\forall \beta$ we have $\alpha \vdash_{Ł} \beta$	α proves everything
α is strongly unsat.	$\forall w$ we have $w(\alpha) = 0$	α is always false
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Added in Proof. The terminology "Strongly unsatisfaible/inconsistent" is not standard. I only used it for ease of exposition. I do not know of a standard terminology for these concepts.

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Equivalent in classical logic by the Deduction Theorem.

Deduction Theorem for CL

For any $\alpha, \beta \in \textsc{Form}$,

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Local Deduction Theorem for Ł

For any $\alpha, \beta \in \textsc{Form}$,

$$\alpha \vdash_{Ł} \beta \quad \text{if, and only if,} \quad \exists n \geqslant 1 \text{ such that } \vdash_{Ł} \alpha^n \to \beta\,.$$

(Recall that $\alpha^n := \underbrace{\alpha \odot \cdots \odot \alpha}_{n \text{ times}}$.)

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Functional completeness

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Is Łukasiewicz logic over 1 variable functionally complete w.r.t. functions $[0,1] \to [0,1]$?

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To make an educated guess at what the answer is, we need to look at more examples. (At the board.)

Notation	**Formal semantics**
$\bot$	$w(\bot) = 0$
$\neg\alpha$	$w(\neg\alpha) = 1 - w(\alpha)$
$\alpha \vee \beta$	$w(\alpha \vee \beta) = \max\{w(\alpha), w(\beta)\}$
$\alpha \wedge \beta$	$w(\alpha \wedge \beta) = \min\{w(\alpha), w(\beta)\}$
$\alpha \oplus \beta$	$w(\alpha \oplus \beta) = \min\{1, w(\alpha) + w(\beta)\}$

Table: Formal semantics of connectives in Łukasiewicz logic.

Definition

A function $f\colon [0,1] \to [0,1]$ is *piecewise linear* if it is continuous, and there is a finite set $\{L_1, \ldots, L_m\}$ of affine linear functions $L_i\colon \mathbb{R} \to \mathbb{R}$, $L_i(x) = a_i x + b_i$ for $a_i, b_i \in \mathbb{R}$, such that, for each $x \in [0,1]$, f agrees with some L_i (depending on x). If such a function is such that each a_i and b_i can be chosen to be integers, then it is called a $\mathbb{Z}$-map.

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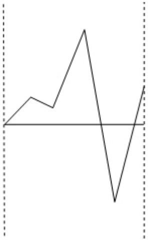

A piecewise linear function $[0,1] \to \mathbb{R}$.

McNaughton's Theorem in 1 variable

A function $f\colon [0,1] \longrightarrow [0,1]$ is definable by a formula in Łukasiewicz logic if, and only if, it is a $\mathbb{Z}$-map.

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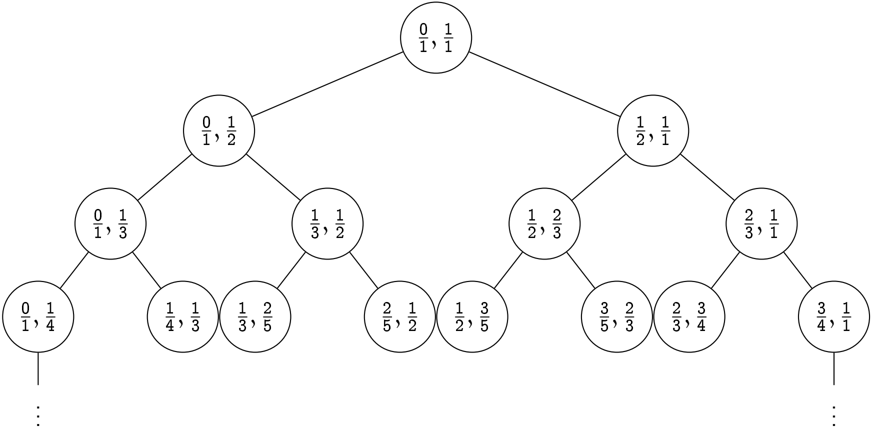
By appropriate generalisation of the notion of $\mathbb{Z}$-maps to functions $[0,1]^\kappa \longrightarrow [0,1]$, the theorem extends to arbitrary sets of variables.



The Farey tree.

$$L_{0,1} = \neg X_1 \qquad R_{0,1} = X_1$$

$$L_{1,1} = L_{0,1} \ominus R_{0,1} \qquad R_{1,1} = R_{0,1} \qquad\qquad L_{1,2} = L_{0,1} \qquad R_{1,2} = R_{0,1} \ominus L_{0,1}$$

$$L_{2,1} = L_{1,1} \ominus R_{1,1} \qquad R_{2,1} = R_{1,1} \qquad L_{2,2} = L_{1,1} \qquad R_{2,2} = R_{1,1} \ominus L_{1,1}$$

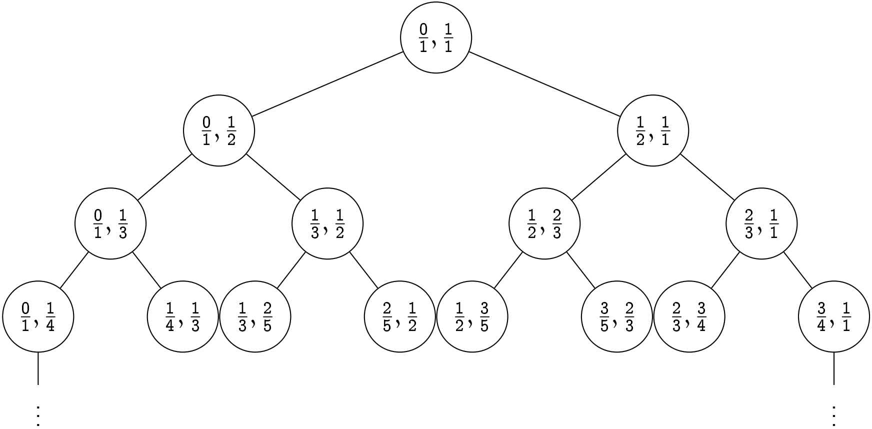
$$L_{2,3} = L_{1,2} \ominus R_{1,2} \qquad R_{2,3} = R_{1,2} \qquad L_{2,4} = L_{1,2} \qquad R_{2,4} = R_{1,2} \ominus L_{1,2}$$

$$L_{3,1}, R_{3,1} \quad L_{3,2}, R_{3,2} \quad L_{3,3}, R_{3,3} \quad L_{3,4}, R_{3,4} \quad L_{3,5}, R_{3,5} \quad L_{3,6}, R_{3,6} \quad L_{3,7}, R_{3,7} \quad L_{3,8}, R_{3,8}$$

$$\vdots \qquad\qquad \vdots$$

The Farey formulæ (related to Schauder bases).

Recall: $\alpha \ominus \beta = \neg(\alpha \to \beta)$; truncated subtraction.



Cauchy's Thm. Every rational number in $(0, 1)$ occurs, automatically in reduced form, as the mediant of the numbers in some node of the Farey tree exactly once. (*Added in proof.* The mediant of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a+c}{b+d}$.)

$$L_{0,1} = \neg X_1 \qquad R_{0,1} = X_1$$

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$$\vdots \qquad \qquad \vdots$$

Thm. Given any maximal consistent theory Θ in Ł (over 1 variable X) that is not just $\{X\}^{\vdash_{Ł}}$ or $\{\neg X\}^{\vdash_{Ł}}$, there is exactly one node in the tree such that there is an integer $n \geqslant 1$ with $\Theta = \{n(L \wedge R)\}^{\vdash_{Ł}}$. Moreover, this n is unique and equals the denominator of the mediant of the node.

Where to learn more

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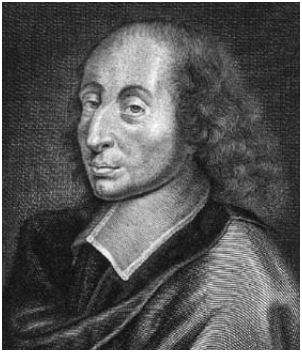
Epilogue: Betting on vague propositions, again

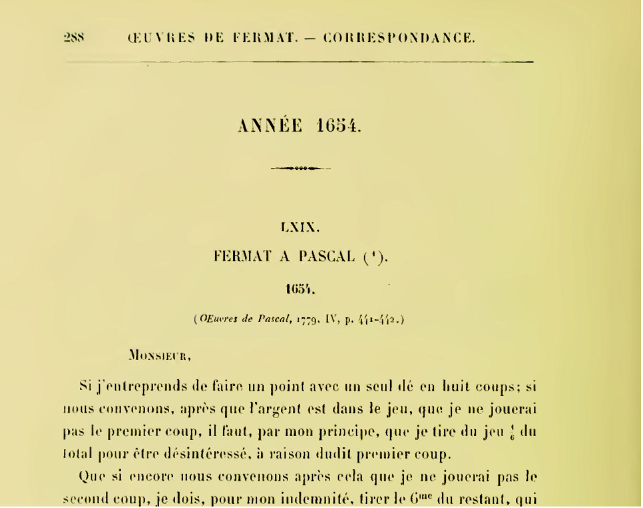
i 

Pierre Fermat (1601 – 1665)



Blaise Pascal (1623 – 1662)

Historiographic cliché: Probability theory begins in 1654, with the correspondence between Fermat and Pascal on the *problem of points, proposed to them by the Chevalier de Méré (born Antoine Gombaud).*



ANNÉE 1654.

LXIX.

FERMAT A PASCAL (1).

1654.

(*Œuvres de Pascal*, 1779, IV, p. 441-442.)

Monsieur,

Si j'entreprends de faire un point avec un seul dé en huit coups; si nous convenons, après que l'argent est dans le jeu, que je ne jouerai pas le premier coup, il faut, par mon principe, que je tire du jeu $\frac{1}{6}$ du total pour être désintéressé, à raison dudit premier coup.

Que si encore nous convenons après cela que je ne jouerai pas le second coup, je dois, pour mon indemnité, tirer le 6me du restant, qui

The Fermat-Pascal correspondence. *Œuvres de Fermat* (ed. Tannery and Henry, Vol. II, p. 288–314, Paris 1894)

Part of Pascal's reply:

2. Votre méthode est très-sûre et est celle qui m'est la première venue à la pensée dans cette recherche ; mais, parce que la peine des combinaisons est excessive, j'en ai trouvé un abrégé et proprement une autre méthode bien plus courte et plus nette, que je voudrois vous pouvoir dire ici en peu de mots : car je voudrois désormais vous ouvrir mon cœur, s'il se pouvoit, tant j'ai de joie de voir notre rencontre. Je vois bien que la vérité est la même à Toulouse et à Paris.

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Neither Pascal nor Fermat explicitly bring logic to bear on probability. What does logic have to do with the theory of probability?

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The set S of all possible outcomes is called the *sample space*.

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An important tradition in the subject regards probability theory as an attempt to model the *possible outcomes of idealised experiments*.
Thus, in the letter from Fermat to Pascal quoted above, the *experiment* is a sequence of eight throws of a die (with faces numbered from 1 to 6).
The *possible outcomes* of the experiment are all possible sequences of points in the eight throws; one such, e.g., is

$$1, 1, 3, 6, 2, 4, 6, 4\ .$$

The set S of all possible outcomes is called the *sample space*.
Certain subsets of S (not necessarily all) are then selected as having special interest for the problem at hand; they form the collection $\mathscr{E}$ of *events.*

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Thus, returning to Fermat's example, the set consisting of the two sequence of points

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corresponds to an event, because it may be described by a proposition. Say,

> *"Either one observes, as the outcome of the experiment, the smallest possible point at each throw, or else one observes the largest possible point at each throw."*

Boole on events vs. propositions:

6. Before we proceed to estimate to what extent known methods may be applied to the solution of problems such as the above, it will be advantageous to notice, that there is another form under which all questions in the theory of probabilities may be viewed; and this form consists in substituting for *events* the propositions which assert that those events have occurred, or will occur; and viewing the element of numerical probability as having reference to the *truth* of those *propositions*, not to the *oc-*

currence of the *events* concerning which they make assertion.

George Boole, *An Investigation of The Laws of Thought*, pp. 247–248, Dublin 1854

Keynes on events vs. propositions:

CH. I FUNDAMENTAL IDEAS 5

4. With the term "event," which has taken hitherto so important a place in the phraseology of the subject, I shall dispense altogether.[1] Writers on Probability have generally dealt with what they term the "happening" of "events." In the problems which they first studied this did not involve much departure from common usage. But these expressions are now used in a way which is vague and ambiguous; and it will be more than a verbal improvement to discuss the truth and the probability of *propositions* instead of the occurrence and the probability of *events*.[2]

John Maynard Keynes, *A Treatise on Probability*, p. 5, Cambridge 1920

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To summarise:

> *The rôle of logic in the theory of probability is to provide a formal model for the notion of event.*

Let us now turn to probabilities.

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Fix a theory Θ. A probability assignment (relative to Θ) is a function

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How do we know that these axioms capture our intuitions about probability (if any)?

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Frank P. Ramsey (1903 – 1930)

Bruno de Finetti (1906 - 1985)

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In case $\sigma_i \geqslant 0$, Blaise hands $\sigma_i \beta(E_i)$ euros to Ada, with the agreement that $\sigma_i w(E_i)$ euros shall be paid back by Ada to Blaise after the occurrence (or lack thereof) of of each event is ascertained.

True to her palindromic name, though, Ada also accepts reverse bets. That is, she also accepts Blaise's negative stakes $\sigma_i < 0$, to the effect that she must hand $|\sigma_i|\beta(E_i)$ euros to Blaise, with the agreement that $|\sigma_i|w(E_i)$ euros shall be paid back by Blaise to Ada in the possible world w.

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Hence, the final balance of Ada's book $\beta\colon \mathscr{E} \to [0,1]$ is given by

$$\sum_{i=1}^{n} (\sigma_i \beta(E_i) - \sigma_i w(E_i)) \ ,$$

where it is understood that money transfers are oriented so that 'positive' means 'Blaise-to-Ada'.

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An assignment of numbers from $[0, 1]$ to the events in $\mathscr{E}$ is coherent if it is not incoherent.

Now we use the Boole-Keynes idea, and we regard the events in $\mathscr{E}$ simply as a family of formulæ in classical logic. We can assume without loss of generality that $\mathscr{E}$ is closed under deduction, i.e. is a theory. Now it makes sense to ask whether an assignment of numbers in $[0,1]$ to $\mathscr{E}$ satisfies Kolmogorov's axioms in the form reviewed above.

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The theorem provides a fundamental operational explanation of Kolmogorov's axioms for finitely additive probabilities.

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The theorem provides a fundamental operational explanation of the axiom for states. It is the beginning of the (nascent) theory of probability of events described by formulæ in a non-classical logic.

Thank you for your attention.