The modal formula (†) $\Box\Diamond p \supset \Box\Diamond\Box\Diamond p$ is not first-order definable

A.B. Karatay*

October 18, 2007

1 Terminology and Notation

By first-order language *corresponding* to a basic modal language I mean a first-order language with identity symbol '$=$', whose only non-logical constant is a binary relation symbol 'R'. Following [1], I use the symbol '$\Vdash$' for the semantic relation in modal logic, and the symbol '$\models$' for the semantic relation in first-order logic.

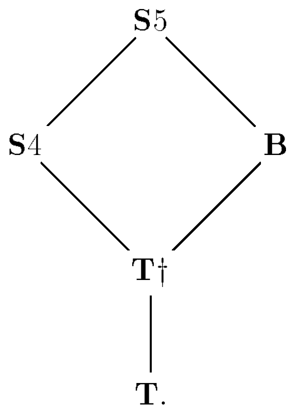
We call a formula α in the basic modal language *first-order definable* if there is a formula ϕ in the corresponding first-order language s.t. for any Kripke frame $\mathcal{F}$, $\mathcal{F} \Vdash \alpha$ iff $\mathcal{F} \models \phi$. Equivalently, α is *first-order definable* if α characterizes an elementary class of structures in the sense of model theory.

Following Chellas [4] I write **T** for the system "**K** + T," **B** for the system "**T** + B," and so on.

*Department of Philosophy, Boğaziçi University, Bebek 34341 İstanbul, Turkey, *karatay@boun.edu.tr*

2 Motivation

The system "T + †" (henceforth **T**†) is placed at a central location in the lattice of the normal systems of modal logic:



The surrounding systems are given in terms of first-order definable modal formulas, so it is a natural question to ask whether † is also first-order definable.[1]

Another motivation comes from Brown's Logic of Ability, [2]. Brown has introduced a formal system, $\mathcal{V}$, aiming to capture the notion of "reliable ability," and defined various "reliable extensions" of this system with sound and complete axiomatizations in parallel to the extension of the modal system **K** to **D**, **T**, and **B**. But a reliable extension that would stand to $\mathcal{V}$ as **S**4 stands to **K** resisted to be given a sound and complete axiomatization. Now, the formula that is expected to yield the desired system has close affinity to our formula †. The hope was that a first-order definition of † would throw light on how to give a sound and complete axiomatization for the required extension of $\mathcal{V}$.

3 Proof

We obtain the result by complicating the construction introduced in [5].

We define a sequence of frames $\mathcal{F}_n = \langle W_n, R_n \rangle$ s.t. $\mathcal{F}_n \models \Box\Diamond p \supset \Box\Diamond\Box\Diamond p$ for all i, $i \geq 1$, but when G is a nonprincipal ultrafilter on N, $\prod \mathcal{F}_n / G \not\models \Box\Diamond p \supset \Box\Diamond\Box\Diamond p$.

[1] I learned about the status of the system **T**† from Prof. Thomas McKay's lectures on Modal Logic. The question of this paper was also posed to me by him. The diagrams are drawn by Paul Taylor's Commutative Diagrams macro.

Put $W_n = \{w_1\} \cup \{w_2\} \cup \{w_{3,1}, \ldots, w_{3,2n+1}\} \cup \{w_{4,1}, \ldots, w_{4,2n+1}\}$ and define R_n on W_n as follows:

$w_1 R_n w_2$;

$w_1 R_n w_{3,i}$, for $1 \leq i \leq 2n+1$;

$w_2 R_n w_{3,i}$, for $1 \leq i \leq 2n+1$;

$w_{3,1} R_n w_{4,1}$ and $w_{3,1} R_n w_{4,2}$;

$w_{3,i} R_n w_{4,i-1}, w_{3,i} R_n w_{4,i+1}$, for $1 < i < 2n+1$;

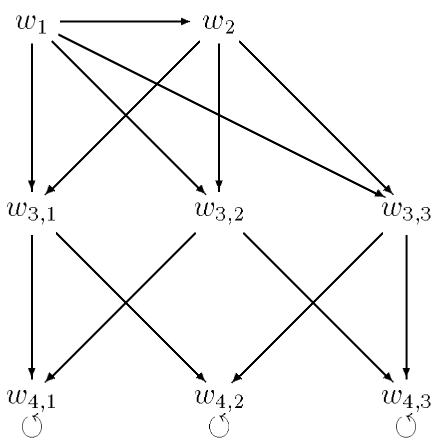
$w_{3,2n+1} R_n w_{4,2n}$ and $w_{3,2n+1} R_n w_{4,2n+1}$;

$w_{4,i} R_n w_{4,i}$, for $1 \leq i \leq 2n+1$.

We shall call a world which is denoted by '$w_{i,j}$' an *ith-level world.*

CLAIM 1: $\mathcal{F}_n \models \Box\Diamond p \supset \Box\Diamond\Box\Diamond p$.

Before starting to prove the claim, it will help to look at a picture of $\mathcal{F}_1$:



Proof. Suppose a frame $\mathcal{F}_n$ is given. We want to show that for every world w in $\mathcal{F}_n$, under every valuation V, we have $V(\Box\Diamond p \supset \Box\Diamond\Box\Diamond p, w) = T$.

Case of $w_{4,i}$, *for* $1 \leq i \leq 2n+1$:

Suppose for a valuation V, $V(\Box\Diamond p, w_{4,i}) = T$. Since the 4th-level worlds are R-related to themselves and only to themselves, we first get $V(\Diamond\Box\Diamond p, w_{4,i}) = T$ and then get $V(\Box\Diamond\Box\Diamond p, w_{4,i}) = T$. Case done.

Case of $w_{3,i}$*, for* $1 < i < 2n+1$:

Suppose for a valuation V, $V(\Box\Diamond p, w_{3,i}) = T$. Then for all j s.t. $w_{3,i} R w_{4,j}$, $V(\Diamond p, w_{4,j}) = T$. But since the 4th level worlds are R-related to themselves and only to themselves, we obtain $V(\Diamond\Box\Diamond p, w_{4,j}) = T$. Hence, $V(\Box\Diamond\Box\Diamond p, w_{3,i}) = T$.

Case of w_2:

$$
\begin{aligned}
V(\Box\Diamond p, w_2) = T &\Rightarrow V(\Diamond p, w_{3,i}) = T \text{ for all } i \\
&\Rightarrow \text{for each } i \text{ there is some } j_i \text{ s.t. } V(p, w_{4,j_i}) = T \\
&\Rightarrow V(\Diamond p, w_{4,j_i}) = T \\
&\Rightarrow V(\Box\Diamond p, w_{4,j_i}) = T \\
&\Rightarrow V(\Diamond\Box\Diamond p, w_{3,i}) = T, \text{ for all } i. \\
&\Rightarrow V(\Box\Diamond\Box\Diamond p, w_2) = T
\end{aligned}
$$

Case of w_1:
Before examining this case we first note the following

LEMMA: *For any modal formula* φ*, for any natural number* $n \geq 1$*, and for any valuation* V*,*
(i) *if* $V(\Diamond\varphi, w_{3,i}) = T$ *for all* i*,* $1 \leq i \leq 2n+1$*, then* $V(\varphi, w_{4,i}) = T$ *in at least* $n+1$ *of the* 4*th-level worlds*; *and*
(ii) *if* $V(\Box\varphi, w_{3,i}) = F$ *for all* i*,* $1 \leq i \leq 2n+1$*, then* $V(\varphi, w_{4,i}) = F$ *in at least* $n+1$ *of the* 4*th-level worlds.*

In order to prove the case of w_1, we'll show that there is no valuation V s.t. $V(\Box\Diamond p, w_1) = T$ and $V(\Box\Diamond\Box\Diamond p, w_1) = F$. Note that

(1) $\qquad V(\Box\Diamond p, w_1) = T \Rightarrow V(\Diamond p, w_{3,i}) = T$ for all i, $1 \leq i \leq 2n+1$.

On the other hand, $V(\Box\Diamond\Box\Diamond p, w_1) = F$ entails that either $V(\Diamond\Box\Diamond p, w_{3,i}) = F$ for some i, $1 \leq i \leq 2n+1$ or $V(\Diamond\Box\Diamond p, w_2) = F$.

Subcase 1. $V(\Box\Diamond p, w_1) = T$ and $V(\Diamond\Box\Diamond p, w_{3,i}) = F$ for some i, $1 \leq i \leq 2n+1$.

$$
\begin{aligned}
V(\Diamond\Box\Diamond p, w_{3,i}) = F \text{ for some } i &\Rightarrow \text{for all } j \text{ s.t. } w_{3,i} R w_{4,j},\ V(\Box\Diamond p, w_{4,j}) = F \\
&\Rightarrow \text{for all } j \text{ s.t. } w_{3,i} R w_{4,j},\ V(\Diamond p, w_{4,j}) = F \\
&\Rightarrow \text{for all } j \text{ s.t. } w_{3,i} R w_{4,j},\ V(p, w_{4,j}) = F \\
&\Rightarrow V(\Diamond p, w_{3,i}) = F.
\end{aligned}
$$

This contradicts with (1) above.

Subcase 2. $V(\Box\Diamond p, w_1) = T$ and $V(\Diamond\Box\Diamond p, w_2) = F$.

By $V(\Box\Diamond p, w_1) = T$ and (1), $V(\Diamond p, w_{3,i}) = T$ for all i, $1 \leq i \leq 2n+1$, and hence by Lemma (i), there are at least $(n+1)$-many i s.t. $V(p, w_{4,i}) = T$.

On the other hand,

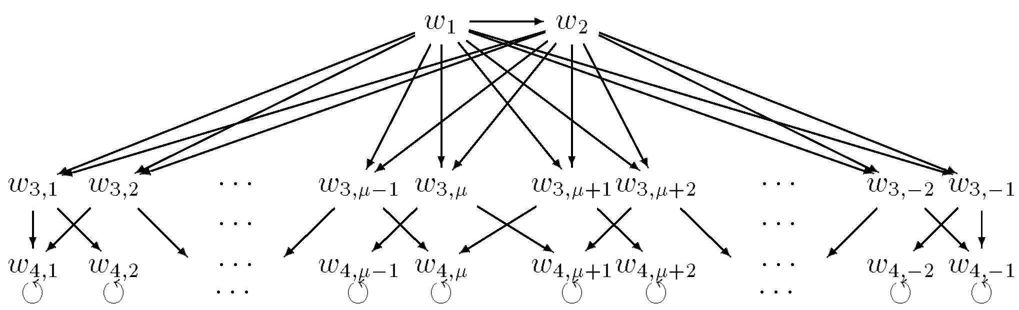
$$\begin{aligned} V(\Diamond\Box\Diamond p, w_2) = F &\Rightarrow V(\Box\Diamond p, w_{3,i}) = F \text{ for all } i,\ 1 \leq i \leq 2n+1 \\ &\Rightarrow V(\Diamond p, w_{4,i}) = F \text{ for at least } n+1\ i \text{ (by Lemma (ii)} \\ &\Rightarrow V(p, w_{4,i}) = F \text{ for at least } n+1\ i \end{aligned}$$

Now, on the one hand, we have p True in at least $n+1$ of the 4th-level worlds, and on the other hand, we have p False in at least $n+1$ of the 4th-level worlds. Since there are altogether $2n+1$ 4th-level worlds, this means that, under the given valuation p is both T and F in at least one 4th-level world. Contradiction.

This completes the proof of Claim 1.

Claim 2: *For a free ultrafilter G on N, $\prod \mathcal{F}_n/G \not\models \Box\Diamond p \supset \Box\Diamond\Box\Diamond p$.*

Proof. By considerations similar to those given in Boolos and Jeffrey [3, pp.194–95] and Łoš's Theorem, the new structure looks as follows:



The order type of the 3rd-level (4th-level) worlds is $\omega + (\omega^* + \omega)\theta + \omega^*$, where θ is some dense linear ordering with no end-points. Let now V be a valuation on $\prod \mathcal{F}_n/G$, subject to the following (partial) initial conditions:

1. $V(p, w_{3,1}) = T$;

2. $V(p, w_{4,1}) = T$, $V(p, w_{4,2})$ and $V(p, w_{4,3})$ are F and $V(p, w_{4,4})$ and $V(p, w_{4,5})$ are T, and so on throughout the initial segment of type ω of the 4th-level worlds;

3. continuing on the middle segment of type $(\omega^* + \omega)\theta$ with $V(p, w_{4,\mu})$ and $V(p, w_{4,\mu+1})$ as T and $V(p, w_{4,\mu+2})$ and $V(p, w_{4,\mu+3})$ as F, and so on;

4. finally, $V(p, w_{4,-1}) = T$, $V(p, w_{4,-2})$ and $V(p, w_{4,-3})$ are F and $V(p, w_{4,-4})$ and $V(p, w_{4,-5})$ are T, and so on throughout the final segment of type ω^*.

Certainly, there are valuations V with the above assignment of truth values to propositional letters. Thus by (2), and the structure of $\prod \mathcal{F}_n/G$, we have $V(\Diamond p, w_{3,\mu}) = T$ for all places μ in the order type $\omega + (\omega^* + \omega)\theta + \omega^*$, and by (1), we have $V(\Diamond p, w_2) = T$, and these two results yield $V(\Box\Diamond p, w_1) = T$. On the other hand, since under the valuation V the truth value assignment to p at the 4th-level worlds has the pattern

$$TFFTTFFTT\ldots\ \ldots TTFFTTFFTTFF\ldots\ \ldots TTFFTTFFT,$$

$V(\Box\Diamond p, w_{3,\mu}) = F$ for each place μ in the order type $\omega + (\omega^* + \omega)\theta + \omega^*$. Hence, $V(\Diamond\Box\Diamond p, w_2) = F$. But then, $V(\Box\Diamond\Box\Diamond p, w_1) = F$, also. Hence $V(\Box\Diamond p \supset \Box\Diamond\Box\Diamond p) = F$ and hence $\prod \mathcal{F}_n/G \not\models \Box\Diamond p \supset \Box\Diamond\Box\Diamond p$.

This completes the proof of the claim 2.

Now, since we have $\{n : \mathcal{F}_n \models \Box\Diamond p \supset \Box\Diamond\Box\Diamond p\} = N \in G$, we have shown that the class of models of the modal formula $\Box\Diamond p \supset \Box\Diamond\Box\Diamond p$ is not closed under ultraproducts. Hence by Corollary 7 in [5], it follows that $\Box\Diamond p \supset \Box\Diamond\Box\Diamond p$ is not first-order definable.

This completes the proof of the main thesis of this paper.

We conclude by providing a

Proof of the Lemma on page 3:

Since for each 3rd-level world there are only two R-accessible 4th-level worlds, by our assumption that $V(\Diamond\varphi, w_{3,i}) = T$ for all i, $1 \le i \le 2n+1$, φ cannot be F at both $w_{4,1}$ and $w_{4,2}$, and neither at both $w_{4,2n+1}$ and $w_{4,2n}$, and for the same reason, φ cannot be F in more than two consecutive worlds—consecutive, that is, with respect to the index i in $w_{4,i}$; in fact, if V assigns the value F to φ at $w_{4,i}$ and $w_{4,i+1}$ then V must assign the value T to φ in at least two consecutive worlds, namely in $w_{4,i+2}$ and $w_{4,i+3}$, for all i, $1 < i < 2n-1$, in order to maintain the condition that $V(\Diamond\varphi, w_{3,i}) = T$ for all i, $1 \le i \le 2n+1$.

Thus, in order to maximize the number of the 4th-level worlds at which V assigns F to φ and minimize the number of the 4th-level worlds at which V

assigns T to φ, we must have at least two consecutive Fs and at most two consecutive Ts alternating throughout.

Now there are two cases when $V(\Diamond\varphi, w_{3,i}) = T$ for all i, $1 \leq i \leq 2n+1$.

Case 1: $V(\Diamond\varphi, w_{4,1}) = T$.

In order to keep the number of worlds at which V assigns T to φ at minimum, let us assume that $V(\varphi, w_{4,2}) = F = V(\varphi, w_{4,3})$. But then this must be followed by at least two consecutive assignments of Ts, which in turn cannot be followed by more than two consecutive assignments of Fs, and so on. Thus we have a finite sequence of Ts and Fs starting with T followed by a number of blocks of the form $FFTT$, finally terminating with a tail of length ≤ 4. The logical possibilities for such sequences are as follows:

$$\begin{array}{lll} T\ FFTT & \cdots & FFTT \\ T\ FFTT & \cdots & FFT \\ T\ FFTT & \cdots & FF \\ T\ FFTT & \cdots & F \end{array}$$

The second and fourth possibilities involve an even number of worlds, so they fall outside the range of our hypothesis that there are $2n+1$-many 3rd level (4th level) worlds.

Since $V(\Diamond p, w_{3,2n+1}) = T$, the third case cannot hold, as we have mentioned at the beginning of this proof. So the only viable possibility is the first one. Since we assume that the length of this sequence is $2n+1$, for some $n \geq 1$, the number of Ts and Fs coming after the initial T is $2n$. Since, moreover, the number of Ts and Fs that follow the initial T are equal, there are exactly n Ts after the initial T. Therefore there are $n+1$ Ts in the first sequence above.

Case 2: $V(\varphi, w_{4,1}) = F$.

Again by definition of R_n and the assumption, $V(\varphi, w_{4,2})$ and $V(\varphi, w_{4,3})$ must be T. According to our min/max strategy, we assume that V assigns F to φ in the next two worlds. The logically possible forms of the sequences of Fs and Ts under this strategy is as follows:

$$\begin{array}{lll} F\ TTFF & \cdots & TTFF \\ F\ TTFF & \cdots & TTF \\ F\ TTFF & \cdots & TT \\ F\ TTFF & \cdots & T \end{array}$$

The first logical possibility cannot hold as we argued for the 3rd line in case 1 above, and the second and fourth possibilities fall outside the range of our hypothesis, since they involve an even number of worlds. There remains the third possibility.

Let k be the number of the blocks of the form $TTFF$ in the third possibility. Then, $4k+3=2n+1$. Hence, $n=2k+1$. Now, since there are k blocks of the form $TTFF$, there are $2k+2$ Ts. But since $2k+1=n$, $2k+2=n+1$. Therefore, there are $n+1$ Ts in the third sequence above.

We see that in either case there are $n+1$ worlds at which V assigns T to p. But recall that we gave the computations for cases 1 and 2 by minimizing the number of worlds at which V assigns T to p. Therefore, under the assumption of the lemma, there are at least $n+1$ 4th level worlds at which V assigns T to φ. This completes the proof of the Lemma (i). A dual argument works for the second part of the Lemma.

An open question: Recall that the McKinsey formula, '$\Box\Diamond p \supset \Diamond\Box p$', is not first-order definable, but Lemmon (and Scott) [6] show that the conjunction of the McKinsey formula with **K** is first-order definable under a certain constraint. The question arises whether such a partial improvement is possible for (†), i.e., whether **T**† is first-order definable under an appropriate constraint.

References

[1] Blackburn, P., M. de Rijke, Y. Venema. *Modal Logic*, Cambridge University Press, 2001.

[2] Brown, M. "On the Logic of Ability." *Journal of Philosophical Logic*, 17, 1–26, 1988.

[3] Boolos, G.S. and R.C. Jeffrey. *Computability and Logic.* Third Edn., Cambridge University Press, 1995.

[4] Chellas, B.F. *Modal Logic.* Cambridge University Press, 1984.

[5] Goldblatt, R.I. "First-Order Definability in Modal Logic." *Journal of Symbolic Logic*, v. 407, 35–40, 1975.

[6] Lemmon, E.J. *An Introduction to Modal Logic.* In collaboration with D. Scott. Ed. by K. Segerberg. Methuen, 1984.