Combinatorial Proofs for the Modal Logic K

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Proof theory is the area of theoretical computer science which studies proofs as mathematical objects. However, unlike many other mathematical fields, proof theory is lacking a representation for its basic objects able to capture the notion of identity. We are used to consider proofs as expressions generated by sets of production rules we call *proof systems*; and the main obstacle to understand when two proofs are the same is this syntactic representation itself. Thus, depending on the chosen formalism, a proof can be represented by different syntactic expressions. Moreover, even in the same proof system, there can not be a "natural way" to identify a *canonical representative*. This condition makes it difficult to understand when two proofs are the same object. As an example we show in Figure 1 a semantic tableau, a resolution proof and a sequent calculus derivation for the same formula.

The standard approach to the question of proof identity is based on rule permutations. Two proofs in the same proof system are considered the to be equal if they can be transformed into each other by a series of simple rule permutation steps. However this can not be considered as a solution since it relies on each specific syntax and, it is not suitable to compare proofs presented in two different proof systems for the same logic.

Combinatorial proofs [6, 7] have been introduced by Hughes to address this problem in classical logic. A combinatorial proof of a formula F consist of a *skew fibration* $f : \mathfrak{C} \mapsto \mathfrak{G}(F)$ between a RB-cograph $\mathfrak{C}$ [9] and the cograph $\mathfrak{G}(F)$ representing the formula F. The notion of cograph [4] and skew fibration [6, 10] are independent from the syntactic restrictions of proof formalisms and are described by graph condition only. Moreover, the correctness of combinatorial proofs can be checked in polynomial time on the size of a proof, i.e. they form a proof system in the sense of Cook and Reckhow [3].

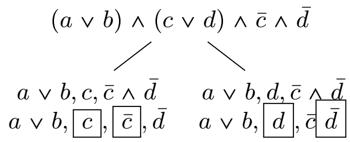
It has been shown in [7, 11, 1] how syntactic proofs in Gentzen sequent calculus, the deep inference system SKS, semantic tableaux, and resolution can be translated into combinatorial proofs. Figure 2 shows the combinatorial proof corresponding to the syntactic proofs in Figure 1.

In this talk we want to address the question whether the theory of combinatorial proofs can be extended to modal logics.

In the literature, proof systems of various kinds have been defined for different modal logics [2, 8, 12, 5]. However, the notion of proof equivalence in modal logic has never been studied. Part of the problem of defining this notion is inheritance of the problem for proof equivalence in classical logic.

We are presently working on the definition of the notion of proof equivalence for different modal logics by means of combinatorial proof. The first step in this investigation is to give a representation of proofs for the modal logic K, for which we show the sequent system LK-K in Figure 3 and the deep infernce system KS-K in Figure 4.

We define a class of cograph, called RG-cograph, suitables to represent formulas with modalities and similarly we extend the notion of RB-cograph which represent the *linear* part of a classical proof, to the one of RGB-cographs.



$$\dfrac{\dfrac{\dfrac{[(a \vee b) \wedge (c \vee d) \wedge \bar{c} \wedge \bar{d}]}{[a \vee b][(c \vee d) \wedge \bar{c} \wedge \bar{d}]}\wedge}{[a \vee b][c \vee d][\bar{c} \wedge \bar{d}]}\wedge}{[a \vee b][\ ]}\, Res^{c \vee d}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash \bar{c}, c}\ \mathsf{AX}}{\vdash \bar{c}, c, d}\ \mathsf{W}}{\vdash (\bar{a} \wedge \bar{b}), \bar{c}, c, d}\ \mathsf{W} \quad \dfrac{\dfrac{\overline{\vdash \bar{d}, d}\ \mathsf{AX}}{\vdash \bar{d}, c, d}\ \mathsf{W}}{\vdash (\bar{a} \wedge \bar{b}), \bar{d}, c, d}\ \mathsf{W}}{\vdash (\bar{a} \wedge \bar{b}), (\bar{c} \wedge \bar{d}), c, d}\ \wedge}{\vdash (\bar{a} \wedge \bar{b}) \vee (\bar{c} \wedge \bar{d}), c, d}\ \vee}{\vdash (\bar{a} \wedge \bar{b}) \vee (\bar{c} \wedge \bar{d}) \vee c, d}\ \vee}{\vdash (\bar{a} \wedge \bar{b}) \vee (\bar{c} \wedge \bar{d}) \vee c \vee d}\ \vee}{}$$

Figure 1: A semantic tableau, a resolution proof and a sequent calculus derivation of $F = (\bar{a} \wedge \bar{b}) \vee (\bar{c} \wedge \bar{d}) \vee c \vee d$

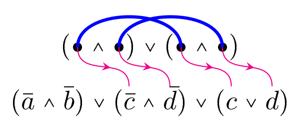


Figure 2: The combinatorial proof corresponing to the proof in Figure 1

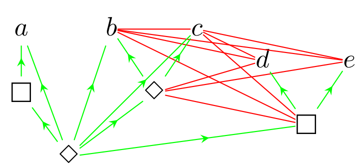
$$\dfrac{}{A, \bar{A}}\,\mathsf{AX} \quad \dfrac{\Gamma, A, B}{\Gamma, A \vee B}\,\vee \quad \dfrac{\Gamma, A \quad B, \Delta}{\Gamma, A \wedge B, \Delta}\,\wedge \quad \dfrac{A, B_1, \ldots, B_n}{\Box A, \Diamond B_1, \ldots, \Diamond B_n}\,\mathsf{K} \quad \dfrac{}{\mathbf{t}}\,\mathbf{t} \quad \Bigg| \quad \dfrac{\Gamma}{\Gamma, A}\,\mathsf{W} \quad \dfrac{\Gamma, A, A}{\Gamma, A}\,\mathsf{C}$$

Figure 3: Sequent system LK-K (cut free) for modal logic K. The first six rules on the left form the sequent system MLLK

$$\mathsf{i}\,\dfrac{\mathbf{t}}{a, \bar{a}} \qquad \mathsf{k}\,\dfrac{\Box(A \vee B)}{\Box A \vee \Diamond B} \qquad \mathsf{s}\,\dfrac{(A \vee B) \wedge C}{A \vee (B \wedge C)} \qquad \mathsf{e}\,\dfrac{\mathbf{t}}{\Box \mathbf{t}} \qquad \mathsf{w}\,\dfrac{\mathbf{f}}{A} \qquad \mathsf{c}\,\dfrac{A \vee A}{A}$$

Figure 4: Deep sequent system KS-K

Below is the RG-cograph of $\Diamond(\Box a \vee (\Diamond(b \wedge c) \wedge \Box(d \vee e)))$:



(1)

For these graphs, we recover a correctness criterion similar to the one given for RB-cographs [9] by means of æ-connectedness and æ-acyclicity (acyclic with respect of alternating paths).

In fact, given a RGB-cograph $\mathfrak{G}(F)$ we are able to define for each $\Box$-node m a set P_m of modality-nodes by means of paths between “same-depth” nodes. Intuitively, each set P_m corresponds to an application of a K-rule. Then we define a RB-cograph $\partial(\mathfrak{G}(F))$ from $\mathfrak{G}(F)$ by transforming each set P_m into a RB-cograph $\partial(P_m)$ and opportunely updating the edges interacting with the nodes with P_m. Thus, a RGB-cograph $\mathfrak{G}(F)$ corresponds to a correct

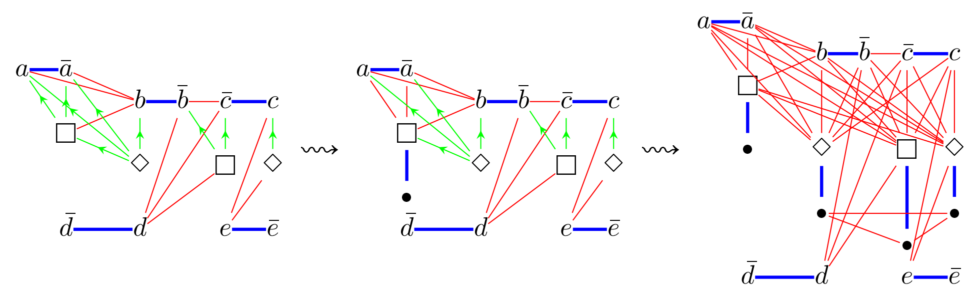


Figure 5: The RGB-cograph $\mathfrak{G}(F)$ of $F = \bar{d} \vee \bar{e} \vee (d \wedge (\bar{b} \wedge \bar{c})) \vee (e \wedge \Diamond c) \vee \Diamond(b \wedge \Box(a \vee \bar{a}))$ and its associated RB-cograph $\partial(\mathfrak{G}(F))$.

derivation if the $\Box$-nodes induce a partition over all modality-nodes and if the RB-cograph $\partial(\mathfrak{G}(F))$ is æ-connected and æ-acyclic.

Using some features of the calculus of structures, we are able to represent K proofs in the deep sequent system KS-K pushing all weakening and contraction rules at the end of a derivation. This allows us to define combinatorial proof by means of axiom-preserving RG-skew fibrations $f : \mathfrak{C} \mapsto \mathfrak{G}(F)$ from a RGB-cograph $\mathfrak{C}$ to the RG-cograph of F.

These results allow us to define a notion of equivalence for proofs in K and give a direct translation of the classical sequent calculus LK-K into combinatorial proofs and vice versa.

References

[1] Matteo Acclavio and Lutz Straßburger. From syntactic proofs to combinatorial proofs. In *International Joint Conference on Automated Reasoning*, pages 481–497. Springer, 2018.

[2] Kai Brünnler. Deep sequent systems for modal logic. *Archive for Mathematical Logic*, 48(6):551–577, 2009.

[3] Stephen A. Cook and Robert A. Reckhow. The relative efficiency of propositional proof systems. *J. of Symb. Logic*, 44(1):36–50, 1979.

[4] R.J Duffin. Topology of series-parallel networks. *Journal of Mathematical Analysis and Applications*, 10(2):303 – 318, 1965.

[5] Melvin Fitting. *Proof methods for modal and intuitionistic logics*, volume 169. Springer Science & Business Media, 1983.

[6] Dominic Hughes. Proofs Without Syntax. *Annals of Math.*, 164(3):1065–1076, 2006.

[7] Dominic Hughes. Towards Hilbert's 24^{th} problem: Combinatorial proof invariants: (preliminary version). *Electr. Notes Theor. Comput. Sci.*, 165:37–63, 2006.

[8] Sonia Marin and Lutz Straßburger. Label-free Modular Systems for Classical and Intuitionistic Modal Logics. In *Advances in Modal Logic 10*, 2014.

[9] Christian Retoré. Handsome proof-nets: perfect matchings and cographs. *Theoretical Computer Science*, 294(3):473–488, 2003.

[10] Lutz Straßburger. A characterization of medial as rewriting rule. In *RTA'07*, volume 4533, pages 344–358. Springer, 2007.

[11] Lutz Straßburger. Combinatorial flows and their normalisation. In Dale Miller, editor, *FSCD'17*, volume 84 of *LIPIcs*, pages 31:1–31:17. Schloss Dagstuhl, 2017.

[12] Heinrich Wansing. Sequent systems for modal logics. In *Handbook of philosophical logic*, pages 61–145. Springer, 2002.