Constructing illoyal algebra-valued models of set theory [*]

Benedikt Löwe[123], Robert Passmann[1], and Sourav Tarafder[4]

[1] Institute for Logic, Language and Computation, Universiteit van Amsterdam
[2] Fachbereich Mathematik, Universität Hamburg
[3] Churchill College, University of Cambridge
[4] Department of Mathematics, St. Xavier's College, Kolkata

Abstract

An algebra-valued model of set theory is called loyal to its algebra if the model and its algebra have the same propositional logic; it is called faithful if all elements of the algebra are truth values of a sentence of the language of set theory in the model. We observe that non-trivial automorphisms of the algebra result in models that are not faithful and apply this to construct three classes of illoyal models: the tail stretches, the transposition twists, and the maximal twists.

The construction of *algebra-valued models of set theory* starts from an algebra $\mathbb{A}$ and a model V of set theory and forms an $\mathbb{A}$-valued model of set theory that reflects both the set theory of V and the logic of $\mathbb{A}$. This construction is the natural generalisation of Boolean-valued models, Heyting-valued models, lattice-valued models, and orthomodular-valued models (Bell, 2011; Grayson, 1979; Ozawa, 2017; Titani, 1999) and was developed by Löwe and Tarafder (2015).

Recently, Passmann (2018) introduced the terms "loyalty" and "faithfulness" while studying the precise relationship between the logic of the algebra $\mathbb{A}$ and the logical phenomena witnessed in the $\mathbb{A}$-valued model of set theory. The model constructed by Löwe and Tarafder (2015) is both loyal and faithful to $\mathbb{PS}_3$.

In this talk, we shall give elementary constructions to produce illoyal models by stretching and twisting Boolean algebras. After we give the basic definitions, we remind the audience of the construction of algebra-valued models of set theory. We then introduce our main technique: a non-trivial automorphisms of an algebra $\mathbb{A}$ excludes values from being truth values of sentences in the $\mathbb{A}$-valued model of set theory. Finally, we apply this technique to produce three classes of models: tail stretches, transposition twists, and maximal twists. This talk is based on Löwe et al. (2018).

1 Basic definitions

Algebras. Let Λ be a set of logical connectives; we shall assume that $\{\wedge, \vee, \mathbf{0}, \mathbf{1}\} \subseteq \Lambda \subseteq \{\wedge, \vee, \rightarrow, \neg, \mathbf{0}, \mathbf{1}\}$. An algebra $\mathbb{A}$ with underlying set A is called a Λ-*algebra* if it has one operation for each of the logical connectives in Λ such that $(A, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a distributive lattice; we can define $\leq$ on $\mathbb{A}$ by $x \leq y$ if and only if $x \wedge y = x$. An element $a \in A$ is an *atom* if it is $\leq$-minimal in $A \backslash \{\mathbf{0}\}$; we write $\mathrm{At}(\mathbb{A})$ for the set of atoms in $\mathbb{A}$. If $\Lambda = \{\wedge, \vee, \rightarrow, \mathbf{0}, \mathbf{1}\}$, we call $\mathbb{A}$ an *implication algebra* and if $\Lambda = \{\wedge, \vee, \rightarrow \neg, \mathbf{0}, \mathbf{1}\}$, we call $\mathbb{A}$ an *implication-negation algebra*.

We call a Λ-algebra $\mathbb{A}$ with underlying set A *complete* if for every $X \subseteq A$, the $\leq$-supremum and $\leq$-infimum exist; in this case, we write $\bigvee X$ and $\bigwedge X$ for these elements of $\mathbb{A}$. A complete Λ-algebra $\mathbb{A}$ is called *atomic* if for every $a \in A$, there is an $X \subseteq \mathrm{At}(\mathbb{A})$ such that $a = \bigvee X$.

Boolean algebras $\mathbb{B} = (B, \wedge, \vee, \neg, \mathbf{0}, \mathbf{1})$ and a *Heyting algebras* $\mathbb{H} = (H, \wedge, \vee, \rightarrow, \mathbf{0}, \mathbf{1})$ are defined as usual.

On an atomic distributive lattice $\mathbb{A} = (A, \wedge, \vee, \mathbf{0}, \mathbf{1})$, we have a canonical definition for a negation operation, the *complementation negation*: since $\mathbb{A}$ is atomic, every element $a \in A$ is uniquely represented by a set $X \subseteq \mathrm{At}(\mathbb{A})$ such that $a = \bigvee X$. Then we define the complementation negation by $\neg_{\mathrm{c}}(\bigvee X) := \bigvee \{t \in \mathrm{At}(\mathbb{A}) \,;\, t \notin X\}$.

Homomorphisms, assignments, & translations. For any two Λ-algebras $\mathbb{A}$ and $\mathbb{B}$, a map $f : A \to B$ is called a Λ-*homomorphism* if it preserves all connectives in Λ; it is called a Λ-*isomorphism* if it is a bijective Λ-homomorphism; isomorphisms from $\mathbb{A}$ to $\mathbb{A}$ are called Λ-*automorphisms*.

Since the propositional formulas $\mathcal{L}_\Lambda$ are generated from the propositional variables P, we can think of any Λ-homomorphism defined on $\mathcal{L}_\Lambda$ as a function on P, homomorphically extended to all of $\mathcal{L}_\Lambda$. If $\mathbb{A}$ is a Λ-algebra with underlying set A, we say that Λ-homomorphisms $\iota : \mathcal{L}_\Lambda \to A$ are $\mathbb{A}$-*assignments*; if S is a set of non-logical symbols, we say that Λ-homomorphisms $T : \mathcal{L}_\Lambda \to \mathrm{Sent}_{\Lambda,S}$ are S-*translations*.

Using assignments, we can define the *propositional logic of* $\mathbb{A}$ as

$$\mathbf{L}(\mathbb{A}) := \{\varphi \in \mathcal{L}_\Lambda \,;\, \iota(\varphi) = \mathbf{1} \text{ for all } \mathbb{A}\text{-assignments } \iota\}.$$

Note that if $\mathbb{B}$ is a Boolean algebra, then $\mathbf{L}(\mathbb{B}) = \mathbf{CPC}$.

Algebra-valued structures and their propositional logic. If $\mathbb{A}$ is a Λ-algebra and S is a set of non-logical symbols, then any Λ-homomorphism $[\![\cdot]\!] : \mathrm{Sent}_{\Lambda,S} \to A$ will be called an $\mathbb{A}$-*valued S-structure*. Note that if $S' \subseteq S$ and $[\![\cdot]\!]$ is an $\mathbb{A}$-valued S-structure, then $[\![\cdot]\!]{\restriction}\mathrm{Sent}_{\Lambda,S'}$ is an $\mathbb{A}$-valued S'-structure. We define the *propositional logic of* $[\![\cdot]\!]$ as

$$\mathbf{L}([\![\cdot]\!]) := \{\varphi \in \mathcal{L}_\Lambda \,;\, [\![T(\varphi)]\!] = \mathbf{1} \text{ for all } S\text{-translations } T\}.$$

Note that if T is an S-translation and $[\![\cdot]\!]$ is an $\mathbb{A}$-valued S-structure, then $\varphi \mapsto [\![T(\varphi)]\!]$ is an $\mathbb{A}$-assignment, so

$$\mathbf{L}(\mathbb{A}) \subseteq \mathbf{L}([\![\cdot]\!]). \tag{$\dagger$}$$

Clearly, $\mathrm{ran}([\![\cdot]\!]) \subseteq A$ is closed under all operations in Λ (since $[\![\cdot]\!]$ is a homomorphism) and thus defines a sub-Λ-algebra $\mathbb{A}_{[\![\cdot]\!]}$ of $\mathbb{A}$. The $\mathbb{A}$-assignments that are of the form $\varphi \mapsto [\![T(\varphi)]\!]$ are exactly the $\mathbb{A}_{[\![\cdot]\!]}$-assignments, so we obtain $\mathbf{L}([\![\cdot]\!]) = \mathbf{L}(\mathbb{A}_{[\![\cdot]\!]})$.

Loyalty & faithfulness. An $\mathbb{A}$-valued S-structure $[\![\cdot]\!]$ is called *loyal to* $\mathbb{A}$ if the converse of ($\dagger$) holds, i.e., $\mathbf{L}(\mathbb{A}) = \mathbf{L}([\![\cdot]\!] = \mathbf{1})$; it is called *faithful to* $\mathbb{A}$ if for every $a \in A$, there is a $\varphi \in \mathrm{Sent}_{\Lambda,S}$ such that $[\![\varphi]\!] = a$; equivalently, if $\mathbb{A}_{[\![\cdot]\!]} = \mathbb{A}$. The two notions central for our paper were introduced by Passmann (2018).

Lemma 1. *If $[\![\cdot]\!]$ is faithful to $\mathbb{A}$, then it is loyal to $\mathbb{A}$.*

Algebra-valued models of set theory. We will work with the general construction of an algebra-valued model of set theory following Löwe and Tarafder (2015), where the precise definitions can be found.

If V is a model of set theory and A is any set, then we construct a universe of *names* $\mathrm{Name}(V, A)$ by transfinite recursion. We then let $S_{V,A}$ be the set of non-logical symbols consisting of the binary relation symbol $\in$ and a constant symbol for every name in $\mathrm{Name}(V, A)$.

If $\mathbb{A}$ is a Λ-algebra with underlying set A, we can now define a map $[\![\cdot]\!]^{\mathbb{A}}$ assigning to each $\varphi \in \mathcal{L}_{\Lambda,S_{V,A}}$ a truth value in $\mathbb{A}$ by recursion, see Löwe and Tarafder (2015) for the precise definitions. As set theorists are usually interested in the restriction to $\mathrm{Sent}_{\Lambda,S}$, we shall use the notation $[\![\cdot]\!]_{\mathbb{A}}$ to refer to this restricted $\mathbb{A}$-valued $\{\in\}$-structure.

The results for algebra-valued models of set theory were originally proved for Boolean algebras, then extended to Heyting algebras:

Theorem 2. *If V is a model of set theory, $\mathbb{B} = (B, \wedge, \vee, \rightarrow, \neg, \mathbf{0}, \mathbf{1})$ is a Boolean algebra or Heyting algebra, and φ is any axiom of ZF, then $[\![\varphi]\!]_{\mathbb{B}} = \mathbf{1}$.*

Lemma 3. *Let $\mathbb{H} = (H, \wedge, \vee, \rightarrow, \mathbf{0}, \mathbf{1})$ be a Heyting algebra and V be a model of set theory. Then $[\![\cdot]\!]_{\mathbb{H}}^{\mathrm{Name}}$ is faithful to $\mathbb{H}$ (and hence, loyal to $\mathbb{H}$).*

Automorphisms and algebra-valued models of set theory. Given a model of set theory V and any Λ-algebras $\mathbb{A}$ and $\mathbb{B}$ and a Λ-homomorphism $f : \mathbb{A} \rightarrow \mathbb{B}$, we can define a map $\widehat{f} :$ $\mathrm{Name}(V, \mathbb{A}) \rightarrow \mathrm{Name}(V, \mathbb{B})$ by $\in$-recursion such that $f([\![\varphi(u_1, \ldots, u_n)]\!]_{\mathbb{A}}) = [\![\varphi(\widehat{f}(u_1), \ldots, \widehat{f}(u_n))]\!]_{\mathbb{B}}$ for all $\varphi \in \mathcal{L}_{\Lambda,\{\in\}}$ with n free variables and $u_1, \ldots, u_n \in \mathrm{Name}(V, \mathbb{A})$. In particular, if $f : \mathbb{A} \rightarrow \mathbb{B}$ is a complete Λ-isomorphism and $\varphi \in \mathrm{Sent}_{\Lambda,\{\in\}}$, then $f([\![\varphi]\!]_{\mathbb{A}}) = [\![\varphi]\!]_{\mathbb{B}}$. Hence, if $f : \mathbb{A} \rightarrow \mathbb{A}$ is a complete Λ-automorphism with $f(a) \neq a$, then there is no $\varphi \in \mathrm{Sent}_{\Lambda,\{\in\}}$ such that $[\![\varphi]\!]_{\mathbb{A}} = a$.

Proposition 4. *If $\mathbb{A} = (A, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is an atomic distributive lattice and $a \in A\backslash\{\mathbf{0}, \mathbf{1}\}$, then there is a $\{\wedge, \vee, \neg_{\mathrm{c}}, \mathbf{0}, \mathbf{1}\}$-automorphism f of $\mathbb{A}$ such that $f(a) \neq a$.*

Note that every $[\![\cdot]\!]_{\mathbb{B}}$ is loyal but not faithful for any non-trivial atomic Boolean algebra $\mathbb{B}$.

2   Stretching and twisting the loyalty of Boolean algebras

In this section, we start from an atomic, complete Boolean algebra $\mathbb{B}$ and modify it, to get an algebra $\mathbb{A}$ that gives rise to an illoyal $[\![\cdot]\!]_{\mathbb{A}}$. The first construction is the well-known construction of tail extensions of Boolean algebras to obtain a Heyting algebra. The other two constructions are *negation twists*: in these, we interpret $\mathbb{B}$ as a Boolean implication algebra via the definition $a \rightarrow b := \neg a \vee b$, and then add a new, twisted negation to it that changes its logic.

What can be considered a negation? When twisting the negation, we need to define a sensible negation. Dunn (1995) lists Hazen's *subminimal negation* as the bottom of his *Kite of Negations*: only the rule of contraposition, i.e., $a \leq b$ implies $\neg b \leq \neg a$, is required. In the following, we shall use this as a necessary requirement to be a reasonable candidate for negation.

Tail stretches Let $\mathbb{B} = (B, \wedge, \vee, \rightarrow, \neg, \mathbf{0}, \mathbf{1})$ be a Boolean algebra and let $\mathbf{1}^* \notin B$ be an additional element that we add to the top of $\mathbb{B}$ to form the *tail stretch* $\mathbb{H}$ as follows: $H :=$ $B \cup \{\mathbf{1}^*\}$, the complete lattice structure of $\mathbb{H}$ is the order sum of $\mathbb{B}$ and the one element lattice $\{\mathbf{1}^*\}$, and $\rightarrow^*$ is defined as follows:[1]

$$a \rightarrow^* b := \left\{ \begin{array}{ll} a \rightarrow b & \text{if } a, b \in B \text{ such that } a \not\leq b, \\ \mathbf{1}^* & \text{if } a, b \in B \text{ with } a \leq b \text{ or if } b = \mathbf{1}^*, \\ b & \text{if } a = \mathbf{1}^*. \end{array} \right.$$

---

[1] In $\mathbb{H}$, we use the (Heyting algebra) definition $\neg_{\mathbb{H}} h := h \rightarrow^* \mathbf{0}$ to define a negation; note that if $\mathbf{0} \neq b \in B$, $\neg_{\mathbb{H}} b = \neg b$, but $\neg_{\mathbb{H}} \mathbf{0} = \mathbf{1}^* \neq \mathbf{1} = \neg \mathbf{0}$.

Transposition twists   Let $\mathbb{B}$ be an atomic Boolean algebra, $a, b \in \mathrm{At}(\mathbb{B})$ with $a \neq b$, and π be the transposition that transposes a and b. We now define a twisted negation by

$$\neg_\pi \left(\bigvee X \right) := \bigvee \{ \pi(t) \in \mathrm{At}(\mathbb{B}) \,;\, t \notin X \}$$

and let the π-*twist* of $\mathbb{B}$ be $\mathbb{B}_\pi := (B, \wedge, \vee, \rightarrow, \neg_\pi, \mathbf{0}, \mathbf{1})$.[2] We observe that the twisted negation $\neg_\pi$ satisfies the rule of contraposition.

Maximal twists   Again, let $\mathbb{B}$ be an atomic Boolean algebra with more than two elements and define the maximal negation by

$$\neg_\mathrm{m} b := \begin{cases} \mathbf{1} & \text{if } b \neq \mathbf{1} \text{ and} \\ \mathbf{0} & \text{if } b = \mathbf{1} \end{cases}$$

for every $b \in B$. We let the *maximal twist* of $\mathbb{B}$ be $\mathbb{B}_\mathrm{m} := (B, \wedge, \vee, \rightarrow, \neg_\mathrm{m}, \mathbf{0}, \mathbf{1})$; once more observe that the maximal negation $\neg_\mathrm{m}$ satisfies the rule of contraposition.

The following is our main result, which is proved by providing non-trivial automorphisms for each of the three constructions.

Theorem 5. *If $\mathbb{B}$ is a Boolean algebra, then its tail stretch, its transposition twist and its maximal twist are not loyal. In particular, the logics of the transposition twist and of the maximal twist is* **CPC**.

References

Bell, J. L. (2011). *Set Theory: Boolean-Valued Models and Independence Proofs*, volume 47 of *Oxford Logic Guides*. Oxford University Press. Third edition.

Dunn, J. M. (1995). Generalized ortho negation. In Wansing, H., editor, *Negation: A Notion in Focus*, pages 3–26. W. De Gruyter.

Grayson, R. J. (1979). Heyting-valued models for intuitionistic set theory. In *Applications of sheaves (Proc. Res. Sympos. Appl. Sheaf Theory to Logic, Algebra and Anal., Univ. Durham, Durham, 1977)*, volume 753 of *Lecture Notes in Math.*, pages 402–414. Springer, Berlin.

Löwe, B., Passmann, R., and Tarafder, S. (2018). Constructing illoyal algebra-valued models of set theory. Submitted.

Löwe, B. and Tarafder, S. (2015). Generalized algebra-valued models of set theory. *Rev. Symb. Log.*, 8(1):192–205.

Ozawa, M. (2017). Orthomodular-valued models for quantum set theory. *Rev. Symb. Log.*, 10(4):782–807.

Passmann, R. (2018). Loyalty and faithfulness of model constructions for constructive set theory. Master's thesis, ILLC, University of Amsterdam.

Titani, S. (1999). A lattice-valued set theory. *Arch. Math. Logic*, 38(6):395–421.

---

[2] Note that we do not twist the implication $\rightarrow$ which remains the implication of the original Boolean algebra $\mathbb{B}$ defined by $x \rightarrow y := \neg_\mathrm{c} x \vee y$.