Logics of skew categorical structures

Tarmo Uustalu, Reykjavik U.
Niccolò Veltri, Tallinn U. of Technology
Cheng-Syuan Wan, Tallinn U. of Technology

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Logics vs categorical structures

- There is a correspondence between *logics* and *categorical structures*, first noticed by Lambek, then further developed by Lawvere, Mann, Szabo, Mints, Soloviev, Dosen and Petrić et al.

(conj-impl) intuit logic	Cartesian closed categories
intuit logic	Cartesian closed categories with finite coproducts
intuit S4	Cartesian closed categories with a lax monoidal comonad
mult intuit linear logic	symm monoidal closed categories
noncomm mult intuit linear logic	monoidal closed categories
Lambek calculus	monoidal biclosed categories

- This is similar to the algebraic logic correspondence of logics and *algebraic structures* as in algebraic logic, but proof-relevant.

- Categorical logic equips a logic with notions of *derivation* (as a opposed to just *consequence*) and *identity of derivations*.

Skew structured categories

- Mult intuit linear logic (the logic of symm monoidal closed categories) drops the structural rules of weakening and contraction of intuitionistic logic:

$$\sout{w : A \Longrightarrow \mathsf{I}}$$
$$\sout{c : A \Longrightarrow A \otimes A}$$

  It is therefore called *substructural* and can be thought of as a *resource* logic rather than a truth logic.

- Recent years have seen the discovery and study of skew monoidal, skew closed and other types of *skew structured* categories by Szlachányi, Street, Bourke, Lack, others.
- These drop one half of unitality and associativity of conjunction:

$$\begin{array}{rr} \lambda : \mathsf{I} \otimes A \Longrightarrow A & \sout{\lambda^{-1} : A \Longrightarrow \mathsf{I} \otimes A} \\ \rho : A \Longrightarrow A \otimes \mathsf{I} & \sout{\rho^{-1} : A \otimes \mathsf{I} \Longrightarrow A} \\ \alpha : (A \otimes B) \otimes C \Longrightarrow A \otimes (B \otimes C) & \sout{\alpha^{-1} : A \otimes (B \otimes C) \Longrightarrow (A \otimes B) \otimes C} \end{array}$$

- Skew structured categories define logics yet more substructural than mult intuit linear logic.

This talk: Skew categorical logic

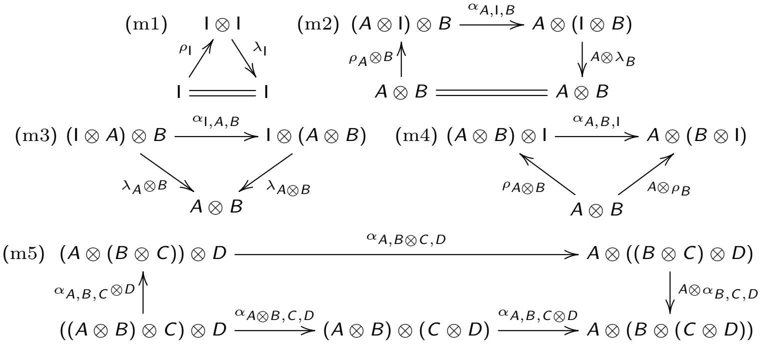
- We have been developing the proof theory of skew structured categories.
- This talk:
  - skew monoidal categories (U., V., Zeilberger, MFPS 2018)
  - skew monoidal closed categories (U., V., W., NCL 2022)
- Other work:
  - partially normal skew monoidal categories (U., V., Zeilberger, ACT 2020)
  - skew closed and skew prounital closed categories (including natural deduction) (U., V., Zeilberger, LFMTP 2020)
  - symmetric skew monoidal categories (V., WoLLIC 2021)
- In progress or stuck:
  - Cartesian skew monoidal categories
  - skew biclosed categories

Monoidal categories

- A *monoidal category* (Bénabou, Mac Lane) is a category $\mathbb{C}$ together with an object I, a functor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and nat. isomorphisms λ, ρ, α with components

$$\lambda_A : \mathsf{I} \otimes A \to A$$
$$\rho_A : A \to A \otimes \mathsf{I}$$
$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

  such that



- Kelly found that (m1), (m3), (m4) follow from (m2), (m5).

Examples

- $(\mathsf{Set}, 1, \times)$ is a monoidal category.
- $(\mathsf{Set}, 0, +)$ is also a monoidal category.

- A preorder is the same as a thin category (at most one map between any two objects).
- A monoid is the same as a discrete monoidal category.
- A preordered monoid is the same as a thin monoidal category.

- A category is a "proof-relevant" generalization of a preordered set.
- A monoidal category is a "proof-relevant" generalization of a preordered monoid.

Coherence

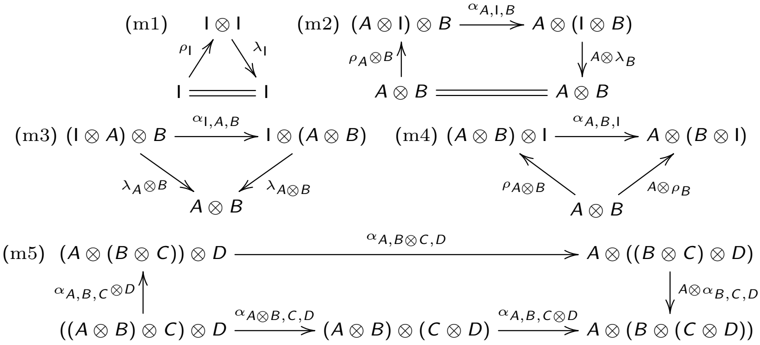
- (Mac Lane) The free monoidal category on a set of objects enjoys a very simple form of (effective) coherence.
    - It is (very easily) decidable if there is a map between two objects A, B, and to exhibit one in this case.
    - Moreover, if there is a map, it is unique.

Skew monoidal categories

- A *skew monoidal category* (Szlachányi) is a category $\mathbb{C}$ together with an object I, a functor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and nat. transfs. λ, ρ, α with components

$$\lambda_A : \mathsf{I} \otimes A \to A$$
$$\rho_A : A \to A \otimes \mathsf{I}$$
$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

such that



- (m1), (m3), (m4) do not follow from (m2), (m5) in this situation.

Examples

- $(\mathsf{Ptd}, 0', +')$ where
  Ptd is the class of pointed sets
  $0' = (1, *)$
  $(X, p) +' (Y, q) = (X + Y, \mathsf{inl}\, p)$
  is a skew monoidal category.

- Given a category $\mathbb{C}$ and a functor $J : \mathbb{J} \to \mathbb{C}$ such that
  $\mathrm{Lan}_J F : \mathbb{C} \to \mathbb{C}$ exists for any $F : \mathbb{J} \to \mathbb{C}$.
  Let $F \cdot^J G = \mathrm{Lan}_J F \cdot G$.
  Then $([\mathbb{J}, \mathbb{C}], J, \cdot^J)$ is a skew monoidal category.

  Relative monads on J are the same as monoids in this skew monoidal category.

Coherence?

- It is not obvious at all when we have zero, one or more maps between two given objects in the free skew monoidal category on a set of objects At or when two given maps between two given objects are the same.

- There are no maps

  $X \to \mathsf{I} \otimes X$,
  $X \otimes \mathsf{I} \to X$,
  $X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$

  for X, Y, Z from At.

- We have distinct maps

  $\rho \circ \lambda \doteq\!\!\!\!/\; \mathrm{id} : \mathsf{I} \otimes \mathsf{I} \to \mathsf{I} \otimes \mathsf{I}$,
  $\mathrm{id} \doteq\!\!\!\!/\; \alpha \circ \rho \otimes \lambda : X \otimes (\mathsf{I} \otimes Y) \to X \otimes (\mathsf{I} \otimes Y)$,
  $\mathrm{id} \doteq\!\!\!\!/\; \rho \otimes \lambda \circ \alpha : (X \otimes \mathsf{I}) \otimes Y \to (X \otimes \mathsf{I}) \otimes Y$.

- This means that the logic of skew monoidal categories is more interesting in comparison to posit mult linear logic—the same consequence can have multiple distinct derivations.

Categorical calculus

- Essentially by definition, the free skew monoidal category on a set At can be *presented* as a deductive system, a "categorical" or Hilbert-style calculus.
- Objects are formulae.
- *Formulae* are atoms $X \in \mathrm{At}$, I and $A \otimes B$ where A, B are formulae.
- Maps are equivalence classes of derivations of sequents $A \Longrightarrow C$ where A, C are (single) formulae.
- *Derivations* are constructed with these inference rules:

$$\frac{}{A \Longrightarrow A}\ \mathrm{id} \qquad \frac{A \Longrightarrow B \quad B \Longrightarrow C}{A \Longrightarrow C}\ \mathrm{comp}$$

$$\frac{A \Longrightarrow C \quad B \Longrightarrow D}{A \otimes B \Longrightarrow C \otimes D}\ \otimes$$

$$\frac{}{\mathsf{I} \otimes A \Longrightarrow A}\ \lambda \qquad \frac{}{A \Longrightarrow A \otimes \mathsf{I}}\ \rho \qquad \frac{}{(A \otimes B) \otimes C \Longrightarrow A \otimes (B \otimes C)}\ \alpha$$

Categorical calculus ctd

- *Equivalence of derivations* is the congruence $\doteq$ induced by the equations

$$\mathrm{id} \circ f \doteq f \qquad f \doteq f \circ \mathrm{id} \qquad (f \circ g) \circ h \doteq f \circ (g \circ h)$$

$$\mathrm{id} \otimes \mathrm{id} \doteq \mathrm{id} \qquad (h \circ f) \otimes (k \circ g) \doteq h \otimes k \circ f \otimes g$$

$$\lambda \circ \mathrm{id} \otimes f \doteq f \circ \lambda$$
$$\rho \circ f \doteq f \otimes \mathrm{id} \circ \rho$$
$$\alpha \circ (f \otimes g) \otimes h \doteq f \otimes (g \otimes h) \circ \alpha$$

$$\lambda \circ \rho \doteq \mathrm{id} \qquad \mathrm{id} \doteq \mathrm{id} \otimes \lambda \circ \alpha \circ \rho \otimes \mathrm{id}$$
$$\lambda \circ \alpha \doteq \lambda \otimes \mathrm{id} \qquad \alpha \circ \rho \doteq \mathrm{id} \otimes \rho$$
$$\alpha \circ \alpha \doteq \mathrm{id} \otimes \alpha \circ \alpha \circ \alpha \otimes \mathrm{id}$$

Sequent calculus

- Here is a cut-free sequent calculus that turns out to correspond to the categorical calculus. (In fact, it is, by definition, a presentation of the free left-representable skew multicategory.)
- *Sequents* now take the form $S \mid \Gamma \longrightarrow C$ where
  - S (stoup) is an optional formula,
  - Γ (context) is a list of formulae,
  - C is a single formula.
- *Derivations* are constructed with these inference rules:

$$\frac{A \mid \Gamma \longrightarrow C}{- \mid A, \Gamma \longrightarrow C}\ \text{pass} \qquad \frac{}{A \mid \ \longrightarrow A}\ \text{ax}$$

$$\frac{- \mid \Gamma \longrightarrow C}{\mathsf{I} \mid \Gamma \longrightarrow C}\ \text{IL} \qquad \frac{}{- \mid \ \longrightarrow \mathsf{I}}\ \text{IR}$$

$$\frac{A \mid B, \Gamma \longrightarrow C}{A \otimes B \mid \Gamma \longrightarrow C}\ \otimes\text{L} \qquad \frac{S \mid \Gamma \longrightarrow A \quad - \mid \Delta \longrightarrow B}{S \mid \Gamma, \Delta \longrightarrow A \otimes B}\ \otimes\text{R}$$

- IL, $\otimes$L only apply in the stoup.
- $\otimes$R sends the stoup formula, if present, to the 1st premise.

Sequent calculus ctd

- *Equivalence of derivations* is the congruence $\doteq$ induced by

$$\dfrac{}{\mathsf{I} \mid \ \longrightarrow \mathsf{I}}\ \mathsf{ax} \qquad \doteq \qquad \dfrac{\dfrac{}{- \mid \mathsf{I}}\ \mathsf{IR}}{\mathsf{I} \mid \ \longrightarrow \mathsf{I}}\ \mathsf{IL}$$

$$\dfrac{}{A \otimes B \mid \ \longrightarrow A \otimes B}\ \mathsf{ax} \qquad \doteq \qquad \dfrac{\dfrac{\dfrac{}{A \mid \ \longrightarrow A}\ \mathsf{ax} \quad \dfrac{\dfrac{}{B \mid \ \longrightarrow B}\ \mathsf{ax}}{- \mid B \longrightarrow B}\ \mathsf{pass}}{A \mid B \longrightarrow A \otimes B}\ \otimes\mathsf{R}}{A \otimes B \mid \ \longrightarrow A \otimes B}\ \otimes\mathsf{L}$$

$$\dfrac{\dfrac{A' \mid \Gamma \longrightarrow A}{- \mid A', \Gamma \longrightarrow A}\ \mathsf{pass} \quad - \mid \Delta \longrightarrow B}{- \mid A', \Gamma, \Delta \longrightarrow A \otimes B}\ \otimes\mathsf{R} \qquad \doteq \qquad \dfrac{\dfrac{A' \mid \Gamma \longrightarrow A \quad - \mid \Delta \longrightarrow B}{A' \mid \Gamma, \Delta \longrightarrow A \otimes B}\ \otimes\mathsf{R}}{- \mid A', \Gamma, \Delta \longrightarrow A \otimes B}\ \mathsf{pass}$$

$$\dfrac{\dfrac{- \mid \Gamma \longrightarrow A}{\mathsf{I} \mid \Gamma \longrightarrow A}\ \mathsf{IL} \quad - \mid \Delta \longrightarrow B}{\mathsf{I} \mid \Gamma, \Delta \longrightarrow A \otimes B}\ \otimes\mathsf{R} \qquad \doteq \qquad \dfrac{\dfrac{- \mid \Gamma \longrightarrow A \quad - \mid \Delta \longrightarrow B}{- \mid \Gamma, \Delta \longrightarrow A \otimes B}\ \otimes\mathsf{R}}{\mathsf{I} \mid \Gamma, \Delta \longrightarrow A \otimes B}\ \mathsf{IL}$$

$$\dfrac{\dfrac{A' \mid B', \Gamma \longrightarrow A}{A' \otimes B' \mid \Gamma \longrightarrow A}\ \otimes\mathsf{L} \quad - \mid \Delta \longrightarrow B}{A' \otimes B' \mid \Gamma, \Delta \longrightarrow A \otimes B}\ \otimes\mathsf{R} \qquad \doteq \qquad \dfrac{\dfrac{A' \mid B', \Gamma \longrightarrow A \quad - \mid \Delta \longrightarrow B}{A' \mid B', \Gamma, \Delta \longrightarrow A \otimes B}\ \otimes\mathsf{R}}{A' \otimes B' \mid \Gamma, \Delta \longrightarrow A \otimes B}\ \otimes\mathsf{L}$$

Categorical calculus vs sequent calculus

- Define

$$\llbracket - \langle\!\langle = \mathsf{I}$$
$$\llbracket A \langle\!\langle = A$$

and

$$A \langle\!\langle \ \rrbracket = A$$
$$A \langle\!\langle B, \Gamma \rrbracket = (A \otimes B) \langle\!\langle \Gamma \rrbracket$$

so $A \langle\!\langle A_1, A_2 \ldots, A_n \rrbracket = (\ldots(A \otimes A_1) \otimes A_2) \ldots) \otimes A_n$.

- There is a bijection between
  - derivations of $\llbracket S \langle\!\langle \ \langle\!\langle \Gamma \rrbracket \Longrightarrow C$ in the categorical calculus (up to $\doteq$) and
  - derivations of $S \mid \Gamma \longrightarrow C$ in the sequent calculus (up to $\overset{\circ}{=}$)

What makes this work?

- We can easily construct derivations to correspond to λ_A, ρ_A, $\alpha_{A,B,C}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \mid \ \longrightarrow A}\,\mathsf{ax}}{- \mid A \longrightarrow A}\,\mathsf{pass}}{\mathsf{I} \mid A \longrightarrow A}\,\mathsf{IL}}{\mathsf{I} \otimes A \mid \ \longrightarrow A}\,\otimes\mathsf{L}$$

$$\dfrac{\dfrac{}{A \mid \ \longrightarrow A}\,\mathsf{ax} \qquad \dfrac{}{- \mid \ \longrightarrow \mathsf{I}}\,\mathsf{IR}}{A \mid \ \longrightarrow A \otimes \mathsf{I}}\,\otimes\mathsf{R}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \mid \ \longrightarrow A}\,\mathsf{ax} \qquad \dfrac{\dfrac{\dfrac{}{B \mid \ \longrightarrow B}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{C \mid \ \longrightarrow C}\,\mathsf{ax}}{- \mid C \longrightarrow C}\,\mathsf{pass}}{B \mid C \longrightarrow B \otimes C}\,\otimes\mathsf{R}}{- \mid B, C \longrightarrow B \otimes C}\,\mathsf{pass}}{A \mid B, C \longrightarrow A \otimes (B \otimes C)}\,\otimes\mathsf{R}}{A \otimes B \mid C \longrightarrow A \otimes (B \otimes C)}\,\otimes\mathsf{L}}{(A \otimes B) \otimes C \mid \ \longrightarrow A \otimes (B \otimes C)}\,\otimes\mathsf{L}$$

What makes this work? ctd

- But we cannot construct derivations for converse sequents for $A = X$, $B = B$, $C = Z$:

$$\frac{X \mid \overset{??}{\mathsf{I}} \longrightarrow X}{X \otimes \mathsf{I} \mid\ \longrightarrow X}\ \otimes\mathsf{L}$$

(we cannot apply IL in the context),

$$\frac{X \mid\ \overset{??}{\longrightarrow} \mathsf{I} \qquad - \mid\ \overset{??}{\longrightarrow} X}{X \mid\ \longrightarrow \mathsf{I} \otimes X}\ \otimes\mathsf{R}$$

(we cannot split the antecedent suitably at $\otimes$R),

$$\frac{\dfrac{X \mid Y \otimes Z \overset{??}{\longrightarrow} X \otimes Y \qquad - \mid\ \overset{??}{\longrightarrow} Z}{X \mid Y \otimes Z \longrightarrow (X \otimes Y) \otimes Z}\ \otimes\mathsf{R}}{X \otimes (Y \otimes Z) \mid\ \longrightarrow (X \otimes Y) \otimes Z}\ \otimes\mathsf{L} \qquad \frac{\dfrac{X \mid\ \overset{??}{\longrightarrow} X \otimes Y \qquad - \mid Y \overset{??}{\otimes} Z \longrightarrow Z}{X \mid Y \otimes Z \longrightarrow (X \otimes Y) \otimes Z}\ \otimes\mathsf{R}}{X \otimes (Y \otimes Z) \mid\ \longrightarrow (X \otimes Y) \otimes Z}\ \otimes\mathsf{L}$$

(we cannot apply $\otimes$L in the context, must therefore apply $\otimes$R first but cannot split the antecedent suitably).

Focused fragment

- The equational theory on sequent calculus derivations is locally confluent and strongly normalizing.
- Normal-form derivations can be described as derivations in a focused fragment.
- The focused calculus has two sequent forms.
  L-*sequents* are $S \mid \Gamma \longrightarrow_{\mathsf{L}} C$ where S is a general stoup.
  R-*sequents* are $T \mid \Gamma \longrightarrow_{\mathsf{R}} C$ where T is an optional atom.
- *Derivations* are constructed with these inference rules:

$$\frac{A \mid \Gamma \longrightarrow_{\mathsf{L}} C}{- \mid A, \Gamma \longrightarrow_{\mathsf{L}} C}\ \mathsf{pass} \qquad \frac{T \mid \Gamma \longrightarrow_{\mathsf{R}} C}{T \mid \Gamma \longrightarrow_{\mathsf{L}} C}\ \mathsf{switch} \qquad \frac{}{X \mid \ \longrightarrow_{\mathsf{R}} X}\ \mathsf{ax}$$

$$\frac{- \mid \Gamma \longrightarrow_{\mathsf{L}} C}{\mathsf{I} \mid \Gamma \longrightarrow_{\mathsf{L}} C}\ \mathsf{IL} \qquad \frac{}{- \mid \ \longrightarrow_{\mathsf{R}} \mathsf{I}}\ \mathsf{IR}$$

$$\frac{A \mid B, \Gamma \longrightarrow_{\mathsf{L}} C}{A \otimes B \mid \Gamma \longrightarrow_{\mathsf{L}} C}\ \otimes\mathsf{L} \qquad \frac{T \mid \Gamma \longrightarrow_{\mathsf{R}} A \quad - \mid \Delta \longrightarrow_{\mathsf{L}} B}{T \mid \Gamma, \Delta \longrightarrow_{\mathsf{R}} A \otimes B}\ \otimes\mathsf{R}$$

- The focused rules define a sound and complete root-first proof search strategy.
- Multiple derivations of an L-sequent result from
  (i) choices between pass and switch and
  (ii) choices between different splits of the context in $\otimes$R.

Sequent calculus vs focused fragment

- There is a bijection between
  - derivations of $S \mid \Gamma \longrightarrow C$ in the sequent calculus (up to $\stackrel{\circ}{=}$) and
  - derivations of $S \mid \Gamma \longrightarrow_{\mathsf{L}} C$ in the focused calculus.

- This gives an (effective) coherence result:
  - To enumerate, without duplicates, all maps $A \to C$ of the free skew monoidal category on At (presented as categorical calculus derivations):

    find all focused derivations of $A \mid \quad \longrightarrow_{\mathsf{L}} C$ and translate those to the categorical calculus.

  - To compare two maps $A \to C$ (presented as categorical calculus derivations) for equality:

    translate them to focused derivations of $A \mid \quad \longrightarrow_{\mathsf{L}} C$ and compare the results.

Skew monoidal closed categories

- A *skew monoidal closed category* is a skew monoidal category $(\mathbb{C}, \mathsf{I}, \otimes, \lambda, \rho, \alpha)$ together with a functor $\multimap : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$ such that

$$- \otimes B \dashv B \multimap -$$

for any object B.

Categorical calculus

- Add formulae $A \multimap B$.

- Add inference rules

$$\frac{A \otimes B \Longrightarrow C}{A \Longrightarrow B \multimap C}\ \pi \qquad \frac{A \Longrightarrow B \multimap C}{A \otimes B \Longrightarrow C}\ \pi^{-1}$$

  and some equations for $\doteq$.

Sequent calculus

- Add formulae $A \multimap B$.

- Add inference rules

$$\frac{- \mid \Gamma \longrightarrow A \quad B \mid \Delta \longrightarrow C}{A \multimap B \mid \Gamma, \Delta \longrightarrow C} \multimap \mathsf{L} \qquad \frac{S \mid \Gamma, A \longrightarrow B}{S \mid \Gamma \longrightarrow A \multimap B} \multimap \mathsf{R}$$

and some equations for $\circeq$.

Focused fragment (a first attempt)

- We need four sequent forms for four phases of proof search:

$$S \mid \Gamma \longrightarrow_{\mathsf{RI}} C \qquad S \mid \Gamma \longrightarrow_{\mathsf{LI}} P \qquad T \mid \Gamma \longrightarrow_{\mathsf{P}} P \qquad T \mid \Gamma \longrightarrow_{\mathsf{F}} P$$

  where S is an unrestricted stoup and C and unrestricted formula, but
  - T is a negative stoup (neither I nor $A \otimes B$) and
  - P is a positive formula (not $A \multimap B$).
- The inference rules are:

(right invertible)

$$\frac{S \mid \Gamma, A \longrightarrow_{\mathsf{RI}} B}{S \mid \Gamma \longrightarrow_{\mathsf{RI}} A \multimap B} \multimap\mathsf{R} \qquad \frac{S \mid \Gamma \longrightarrow_{\mathsf{LI}} P}{S \mid \Gamma \longrightarrow_{\mathsf{RI}} P} \mathsf{LI2RI}$$

(left invertible)

$$\frac{- \mid \Gamma \longrightarrow_{\mathsf{LI}} P}{\mathsf{I} \mid \Gamma \longrightarrow_{\mathsf{LI}} P} \mathsf{IL} \qquad \frac{A \mid B, \Gamma \longrightarrow_{\mathsf{LI}} P}{A \otimes B \mid \Gamma \longrightarrow_{\mathsf{LI}} P} \otimes\mathsf{L} \qquad \frac{T \mid \Gamma \longrightarrow_{\mathsf{P}} P}{T \mid \Gamma \longrightarrow_{\mathsf{LI}} P} \mathsf{P2LI}$$

(passivation)

$$\frac{A \mid \Gamma \longrightarrow_{\mathsf{LI}} P}{- \mid A, \Gamma \longrightarrow_{\mathsf{P}} P} \mathsf{pass} \qquad \frac{T \mid \Gamma \longrightarrow_{\mathsf{F}} P}{T \mid \Gamma \longrightarrow_{\mathsf{P}} P} \mathsf{F2P}$$

(focusing)

$$\frac{}{X \mid \ \longrightarrow_{\mathsf{F}} X} \mathsf{ax} \qquad \frac{}{- \mid \ \longrightarrow_{\mathsf{F}} \mathsf{I}} \mathsf{IR} \qquad \frac{T \mid \Gamma \longrightarrow_{\mathsf{RI}} A \quad - \mid \Delta \longrightarrow_{\mathsf{RI}} B}{T \mid \Gamma, \Delta \longrightarrow_{\mathsf{F}} A \otimes B} \otimes\mathsf{R}$$

$$\frac{- \mid \Gamma \longrightarrow_{\mathsf{RI}} A \quad B \mid \Delta \longrightarrow_{\mathsf{LI}} P}{A \multimap B \mid \Gamma, \Delta \longrightarrow_{\mathsf{F}} P} \multimap\mathsf{L}$$

Focused fragment (good version)

- There is too much nondeterminism between $\otimes$R and $\multimap$L as compared to what $\mathrel{\hat{=}}$ allows.

- We could try to order $\otimes$R and $\multimap$L in separate phases, but this does not work: sometimes $\otimes$R needs to be used first, sometimes $\multimap$L.

- We need to keep them in the same phase.
- But we can allow $\multimap$L to be applied after $\otimes$R only if the same application cannot be simulated with applying $\multimap$L first.
- Ie, apply $\multimap$L before $\otimes$R except when it is justified to do it after.
- This requires some *bookkeeping* added to the inference rules.

- There is also too much nondeterminism between $\otimes$R and pass.
- This can be eliminated by similar prioritization of pass over $\otimes$R with the same bookkeeping mechanism.

Takeaway

- Logic and category theory are mutually enriching, especially at their intersection, in categorical proof theory.
  - category theory supplies well-motivated notions of derivation and identity of derivations
  - proof theory helps in stating and proving coherence theorems

- Skew logics are very interesting both logically and category-theoretically.
- In particular, they cast light on the "anatomy" of stronger logics.