Coordinatising Affine Spatial Logics

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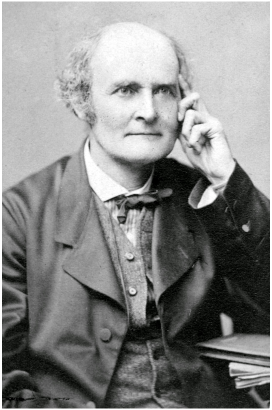
The Beginnings

The general result is as follows [...] assuming [...] a conic [...], we may by means of this conic, by descriptive constructions, divide any line [...] into an infinite series of infnitesimal elements, which are (as a defnition of distance) assumed to be equal; the number of elements between two points [...] measures the distance between the two points [...].



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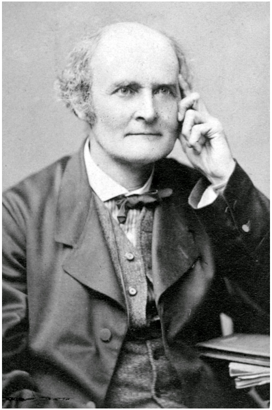

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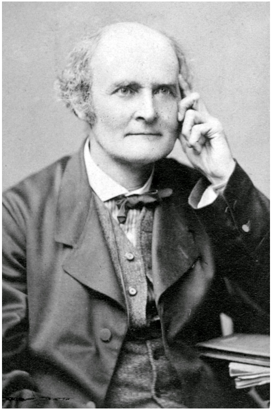

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- Moritz Pasch, a German mathematician
- empirical, natural geometry with an observer at centre
- non-numerical ordering relation of betweenness important

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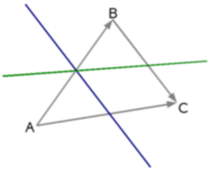

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Enter Philosophy

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- *An Essay on the Foundations of Geometry* (1897)
- *The Principles of Mathematics* (1903)

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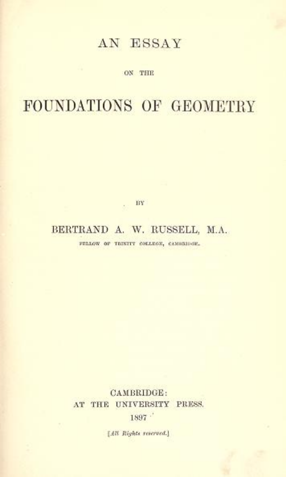
AN ESSAY

ON THE

FOUNDATIONS OF GEOMETRY

BY

BERTRAND A. W. RUSSELL, M.A.

FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

CAMBRIDGE:

AT THE UNIVERSITY PRESS.

1897

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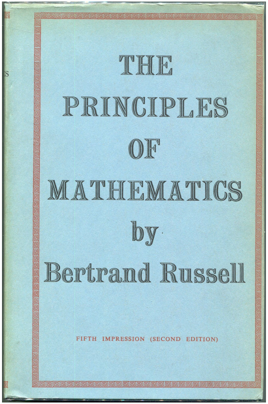

Russell on Geometry

- About Cayley: "He showed that, with the ordinary notion of distance, it can be rendered projective [...]. Not content with this, he suggested a new definition of distance [...]; with this definition, the properties usually known as metrical become projective [...]." (FoG)
- About Pasch: "The present subject [i.e. descriptive geometry] is admirably set forth by Pasch [...] with whose empirical pseudo-philosophical reasons for preferring it to projective Geometry, however, I by no means agree." (PoM)

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IMPORTANCE OF RUSSELL

- Russell took the mathematical work and turned it into a philosophical argument
- qualitative, descriptive geometry is the most important, primary one
- this geometry is nowadays known as affine

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What is Affine Geometry?

Definition

An (n-dimensional) affine transformation of $\mathbb{R}^n$ is a function $\tau : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$\tau(x) = \mathbf{A}x + b,$$

where $\mathbf{A}$ is an invertible $n \times n$ matrix and $b \in \mathbb{R}^n$.

We say that two regions are affine-equivalent if there is an affine transformation from one region to another (this notion naturally extends to sequences of regions). Properties unchanged under affine tranformations are called affine-invariant.

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Affine Transformations ctd.

Theorem

An affine transformation maps straight lines to straight lines, preserves parallelism and ratios of lengths along parallel straight lines. The set of affine transformations forms a group under the operation of composition of functions.

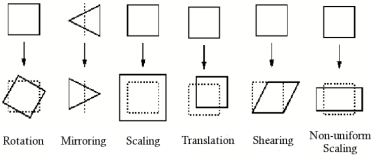

Image source: J.P. de Vries, *Object Recognition: A Shape-Based Approach Using Artificial Neural Networks*

Affine properties

Definition

A set $S \in \mathbb{R}^n$ is called *convex* if for all $\lambda_1, \lambda_2 \in \mathbb{R}$, such that $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$ and for all $x \in S$,

$$\lambda_1 x + \lambda_2 y \in S.$$

Convexity is an affine-invariant property. (So is set inclusion.)

Affine Geometry, Logic and Philosophy

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- *The Axioms of Descriptive Geometry*
- such ideas were developed further in 1970s

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THE AXIOMS OF
DESCRIPTIVE GEOMETRY

by

A. N. WHITEHEAD, Sc.D., F.R.S.
Fellow of Trinity College

HAFNER PUBLISHING CO.
NEW YORK

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THE NEW BEGINNINGS

- more that 100 years after the publication of Russell's *The Foundations of Geometry*, a group of CS researchers took interest in a similar approach to geometry
- the field they established began to be known as Qualitative Spatial Reasoning
- the emphasis was on formalising and analysing commonsensical, non-numerical part of geometry
- the hope: to mimic human-like spatial reasoning in formal settings

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The Focal Points

- regions rather than points became the primitive entities
- a number of logical formalisms were proposed involving qualitative relations among regions
- Region Connection Calculus and its derivative RCC8 are perhaps the best-known ones
- an attempt has been made to connect to previous developments, including Whitehead's ideas

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Few Bones to Pick

- Russell's contribution was not recognized
- the entire area focused on logical systems with topological interpretations
- only a smalll portion of research devoted to affine spatial logics

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Affine Spatial Logics — the Setup

Definition

Let S be a subset of some topological space. We denote the interior of S by S^0 and the closure of S by S^-. S is called regular open if $S = (S)^{-0}$.

The following result is standard.

Proposition

The set of regular open sets in X forms a Boolean algebra $RO(X)$ with top and bottom defined by $1 = X$ and $0 = \emptyset$, and Boolean operations defined by $a \cdot b = a \cap b$, $a + b = (a \cup b)^{-0}$ and $-a = (X \setminus a)^0$.

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What Counts as a Region?

- by a regular open rational polygon we mean a Boolean combination in $RO(\mathbb{R}^2)$ of finitely many half-planes bounded by lines with rational coefficients in $\mathbb{R}^2$.
- we denote the set of all regular open rational polygons in $\mathbb{R}^2$ by $ROQ(\mathbb{R}^2)$.
- $ROQ(\mathbb{R}^2)$ is a Boolean subalgebra of $RO(\mathbb{R}^2)$.
- the notion of regular open rational polygon can be easily extended to that of a polytope, when considering dimensions greater than 2. In general, we write $ROQ(\mathbb{R}^n)$, $n \in \mathbb{N}$, to denote the set of all regular open rational polytopes of dimension n.

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First-Order Affine Spatial Logics

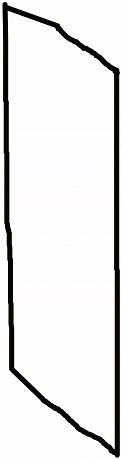
We are interested in structures defined as follows:

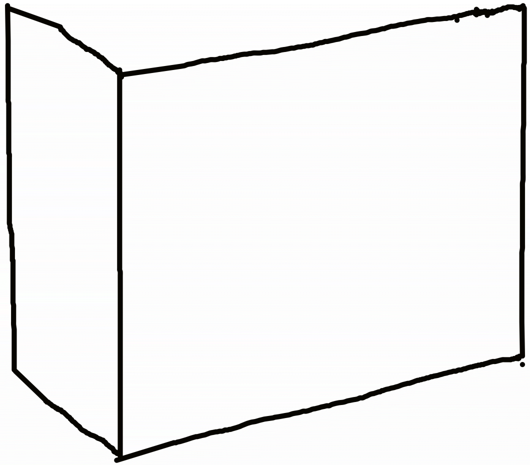
Definition

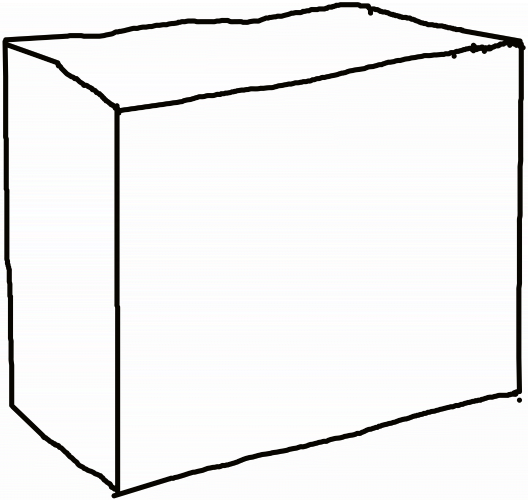
Let $\mathfrak{M}^n = \langle ROQ(\mathbb{R}^n), \mathfrak{conv}^{\mathfrak{M}}, \leq^{\mathfrak{M}} \rangle$, where

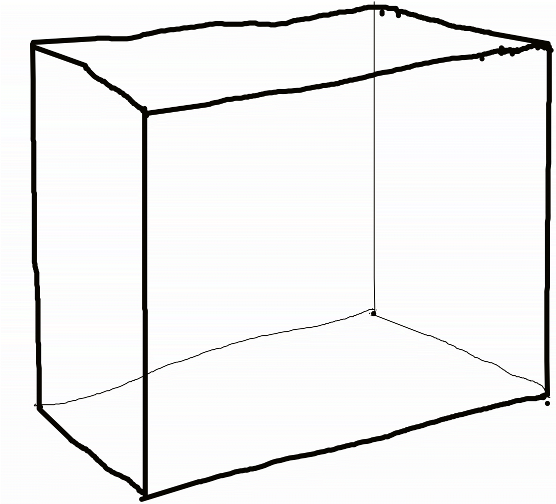
$\leq^{\mathfrak{M}} = \{\langle a, b\rangle \in ROQ(\mathbb{R}^n) \times ROQ(\mathbb{R}^n) \mid a \subseteq b\}$;
$\mathfrak{conv}^{\mathfrak{M}} = \{a \in ROQ(\mathbb{R}^n) \mid a \text{ is convex}\}$.

$$\mathfrak{M}^3 = \langle ROQ(\mathbb{R}^3), \mathfrak{conv}^{\mathfrak{M}}, \leq^{\mathfrak{M}} \rangle$$



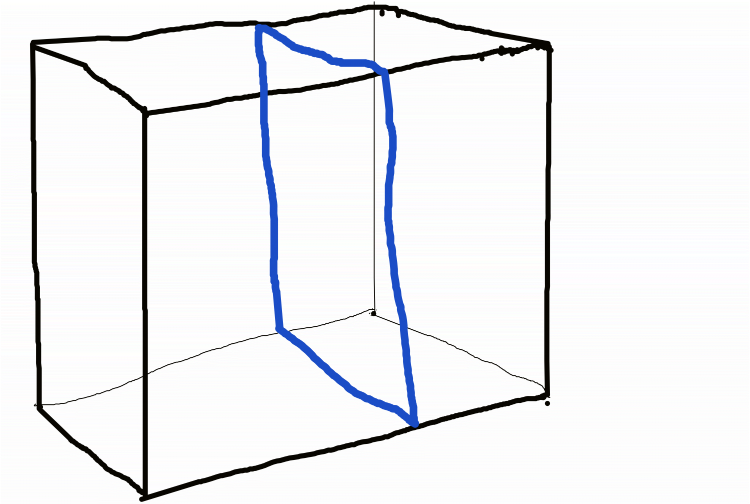


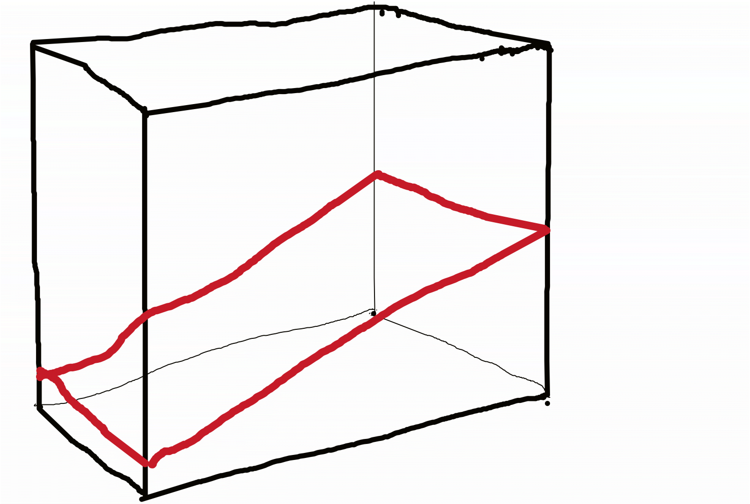


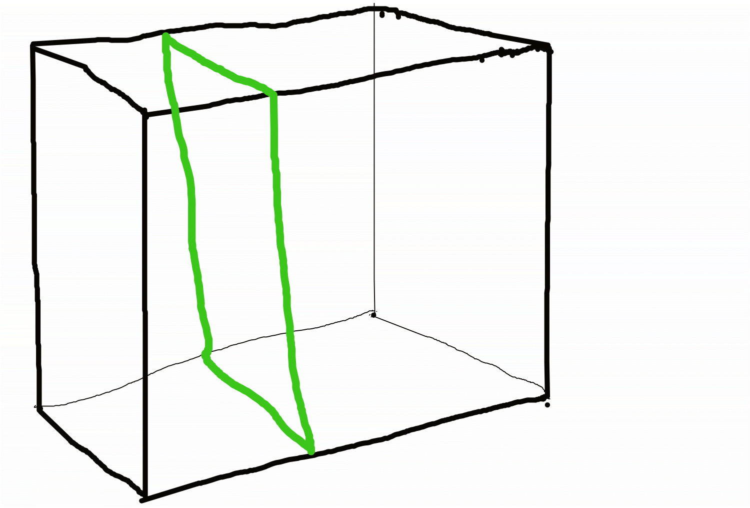


$$\mathfrak{conv}(x) \wedge \mathfrak{conv}(-x)$$

$$\mathfrak{hs}_n(x_1, \ldots, x_n) := \bigwedge_{1 \leq i \leq n} \mathfrak{conv}(x_i) \wedge \mathfrak{conv}(-x_i) \wedge \bigwedge_{\substack{1 \leq i \leq n, \\ 1 \leq j \leq n, \\ i \neq j}} x_i \neq x_j \wedge x_i \neq -x_j$$

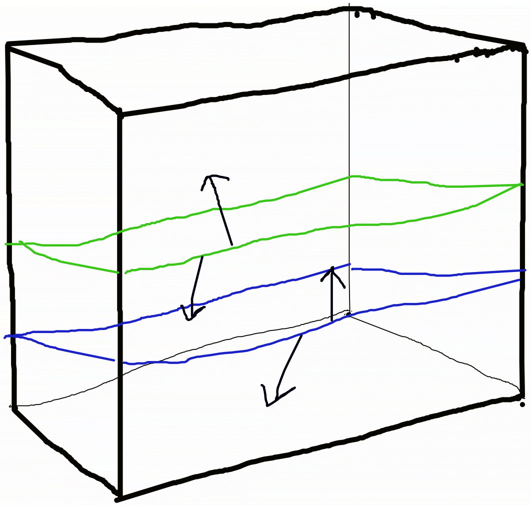


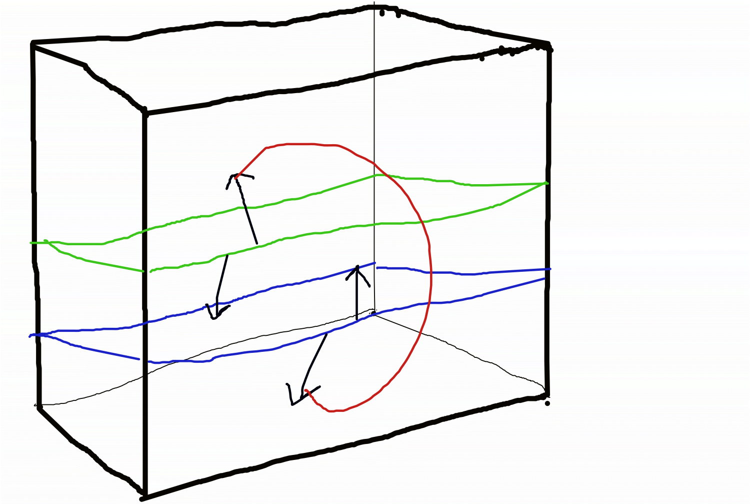


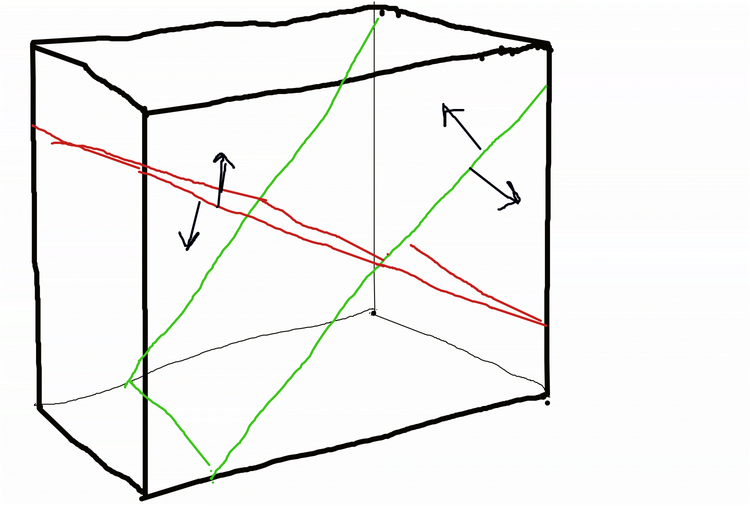


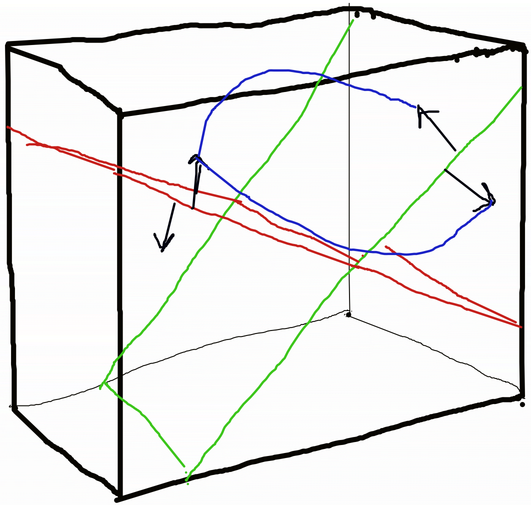
$$\mathfrak{hs}_2(x,y) \wedge ((x \cdot y = 0 \vee x \cdot -y = 0) \vee (-x \cdot y = 0 \vee -x \cdot -y = 0))$$

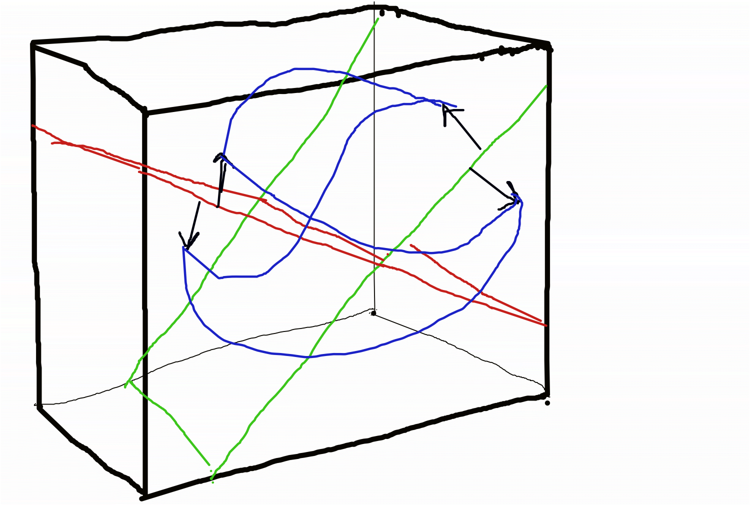
$$\mathfrak{hs}_2(x,y) \wedge \neg((x \cdot y = 0 \vee x \cdot -y = 0) \vee (-x \cdot y = 0 \vee -x \cdot -y = 0))$$





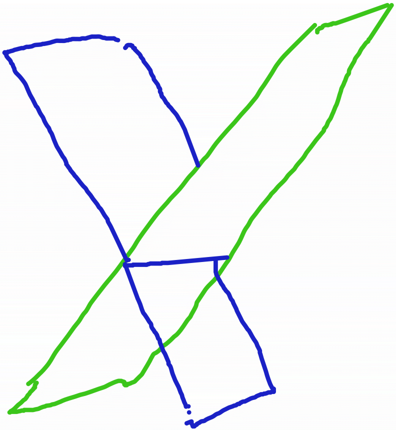


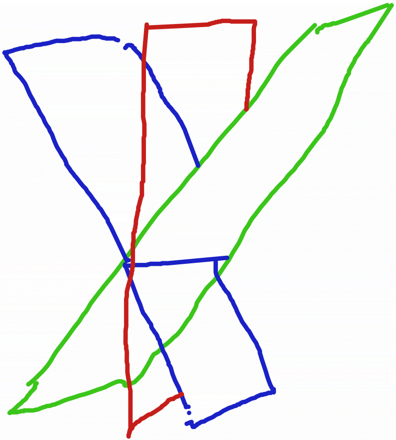


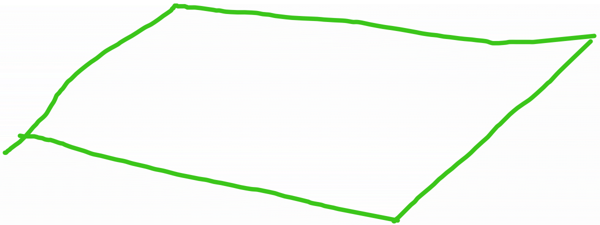


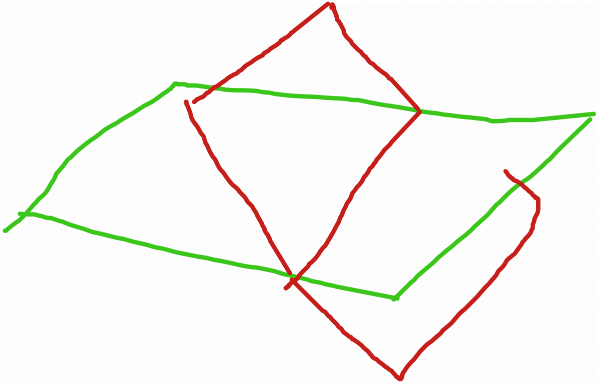
$$\mathfrak{line}(y_1, y_2) \wedge \mathfrak{line}(y_1, y_3) \wedge \mathfrak{line}(y_2, y_3)$$

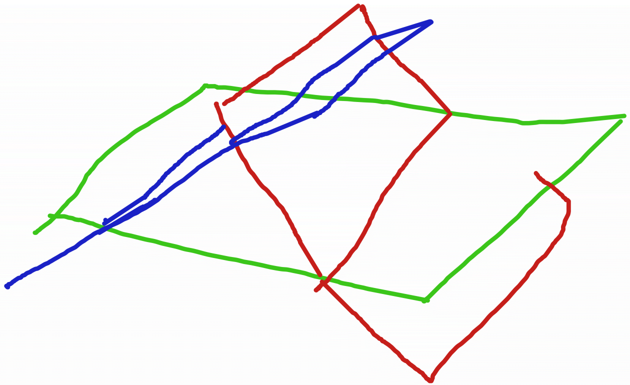
1. a sheaf 2. a prism 3. a corner

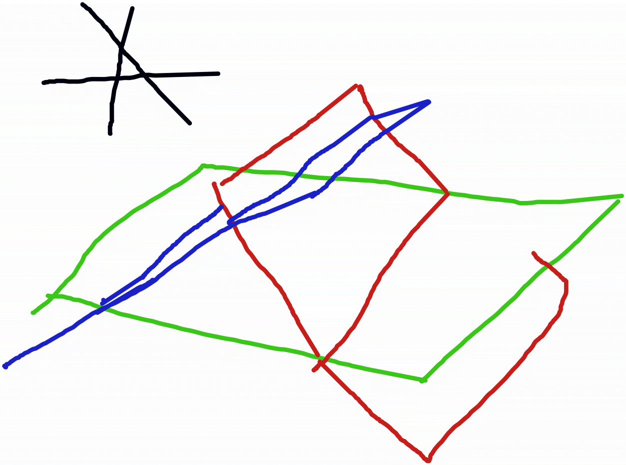


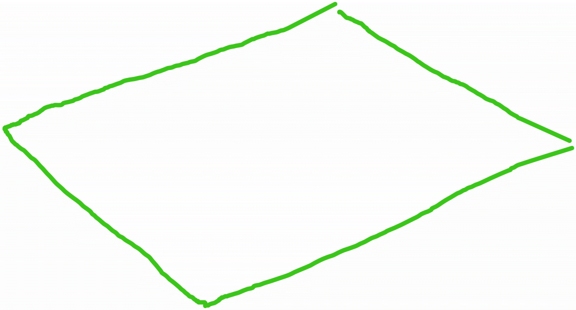


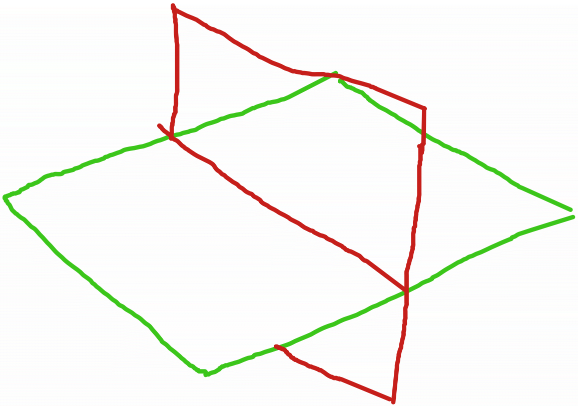


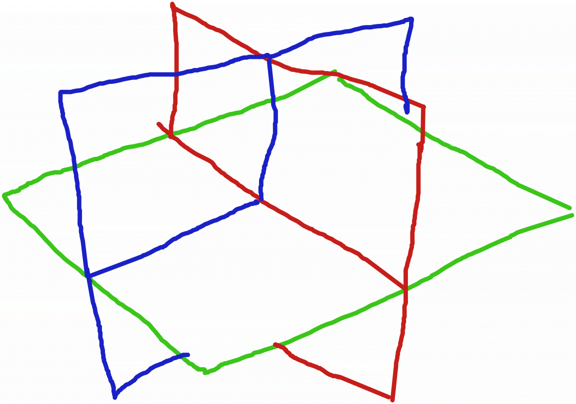












$$\neg\exists x \neg\exists y \neg\exists z(((x = y_1 \vee x = -y_1) \wedge (y = y_2 \vee y = -y_2) \wedge (z = y_3 \vee z = -y$$

$$\wedge (x \cdot y \cdot z = 0))$$

$\mathfrak{corner}(x, y, z)$

$$\mathfrak{frame}(y_1, y_2, y_3, y') :=$$

$$\mathfrak{corner}(y_1, y_2, y_3) \wedge \mathfrak{line}(y_1, y') \wedge \mathfrak{line}(y_2, y') \wedge \mathfrak{line}(y_3, y')$$

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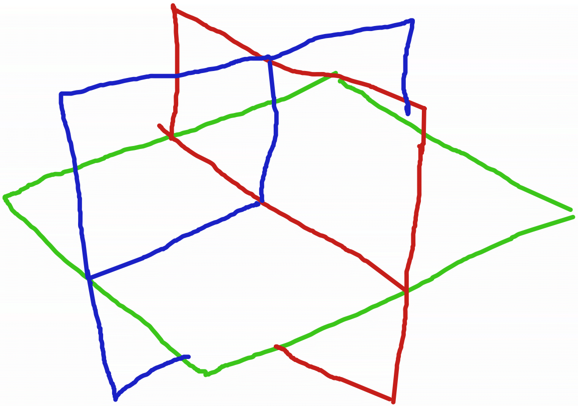

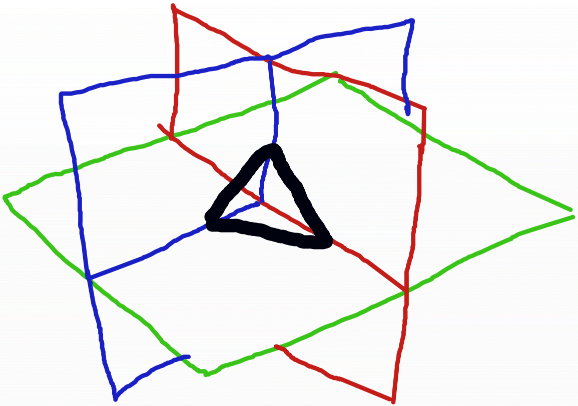

$$\overline{OA} + \overline{OB} = \overline{OC}$$

$$\overline{OA} \cdot \overline{OB} = \overline{OC}$$

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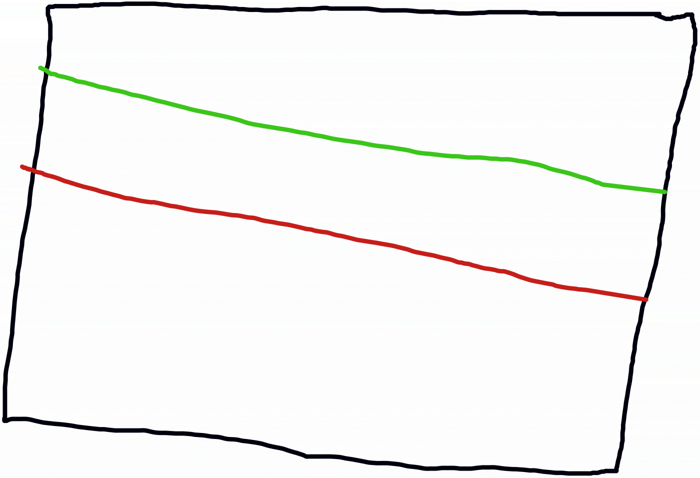

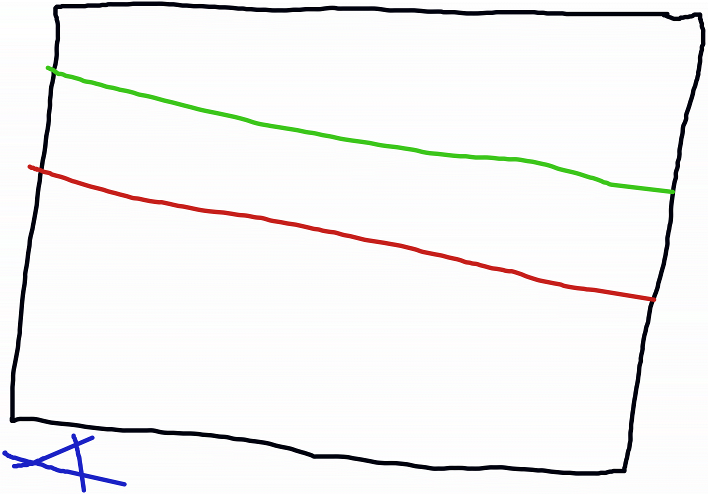

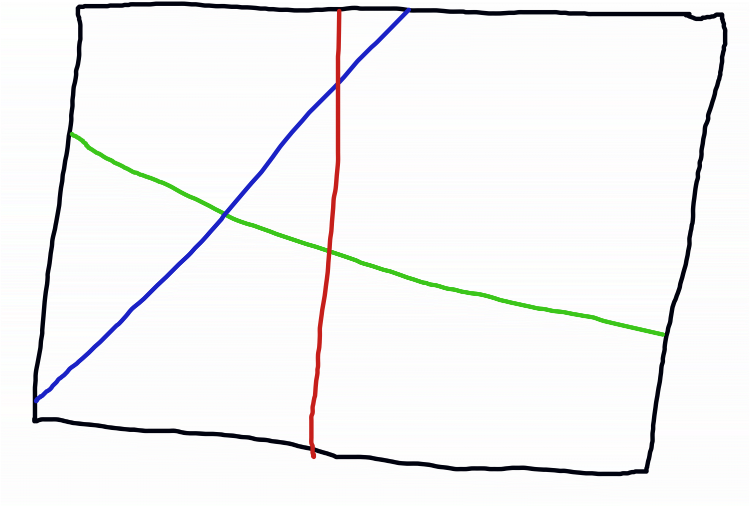

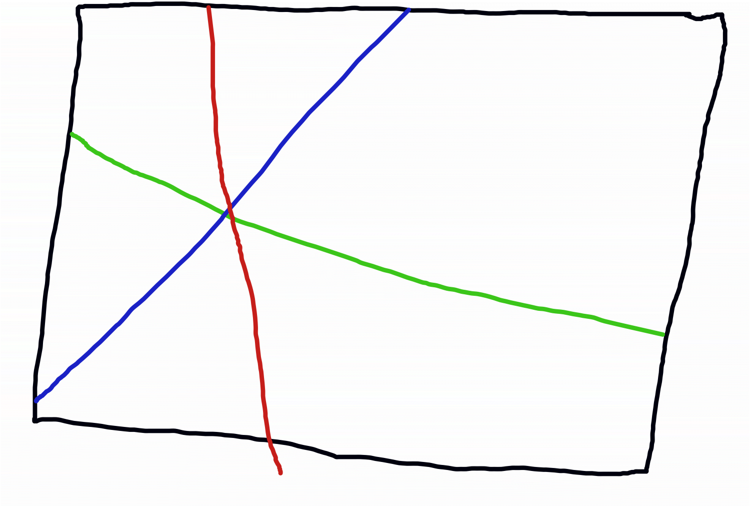

THEOREM

Assuming the coordinate frame setup above and all the introduced shorthands, let m be a line crossing one of the axes at a point $\mathbf{M}$*. Then there exists a formula satisfiable in* $\mathfrak{M}^3$ *if and only if* $\overline{\mathbf{OM}} = n\overline{\mathbf{OI}}$, $n \in \mathbb{Q}$.

THEOREM

Fix a tuple satisfying the $\mathfrak{frame}(y_1, y_2, y_3, y')$ *formula. Let* $h, h' \in ROQ(\mathbb{R}^3)$ *be half-spaces. Then there is a formula* ϕ *satisfiable in* $\mathfrak{M}^3$*, such that (1)* h *satisfies this formula and (2) if* h' *satisfies the formula, then* $h' = h$.

THEOREM

Every $r \in ROQ(\mathbb{R}^3)$ satisfies an affine-complete formula in $\mathfrak{M}^3$.

Final Thoughts

- affine geometry remains closer to our every-day experiences than projective geometry (or topology for that matter) and still it retains the status of non-qualitative
- despite being singled out as an independent entity relatively recently, affine geometry seems to have played an important role since the early days of formal geometry
- finally, affine spatial logics proved to be really expressive, more so than other spatial systems
- the next step: axiomatising the three-dimensional affine spatial logic

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Thank you

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