Associativity of deduction composition in natural deduction

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Introduction and Motivation

Aim

- Kosta Došen argued in his papers *Inferential Semantics* (2015) and *On the Paths of Categories* (2016) that the propositions as types paradigm is less suited for general proof theory because – unlike categorial proof theory based – it makes prominent (categorical) proofs over (hypothetical) inferences
- One specific instance of this, Došen points out, is that the Curry-Howard isomorphism makes the associativity of deduction composition invisible. I will argue that this is not necessarily the case [Pezlar, 2020]

Introduction and Motivation

Motivating quote 1

> The typed lambda coding of the Curry-Howard correspondence [. . .] and the categorial coding in cartesian closed categories are equivalent in a very precise sense. [. . .]. The import of the two formalisms is however not exactly the same. The typed lambda calculus suggests something different about the subject matter than category theory. It makes prominent the *proofs* $t : B$—and we think immediately of the categorical ones, without hypotheses—while category theory is about the *inferences* $f : A \vdash B$. [Došen, 2015]

Introduction and Motivation

Motivating quote 2

> . . . [I]n the Curry-Howard correspondence, one designates deductions by typed lambda terms, which is congenial with understanding proofs in the categorical, and not the hypothetical, i.e. categorial, way [. . .], then composition of deductions is represented by substitution. With that, the associativity of composition becomes invisible, unless one introduces, as it is sometimes done, an explicit substitution operator. [Došen, 2016]

Preliminary Notes

Outline

1 Introduction and Motivation

2 Preliminary Notes

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Preliminary Notes

- general proof theory
- propositions as types principle
- categorial proof theory
- composition of deductions
- associativity

Preliminary Notes

General proof theory

- general proof theory (vs. reductive theory):

> ... proofs are studied in their own right in the hope of understanding their nature ... [Prawitz, 1972]

> Proofs and their representations by formal derivations are treated as principal objects of study, not as mere tools for analyzing the consequence relation. [Kreisel, 1971]

Preliminary Notes

Propositions as types

- see [Curry and Feys, 1958], [Howard, 1980], [De Bruijn, 1968]
- a proposition as the collection (type) of its proofs
- proving a proposition as inhabiting a type
- proofs as programs
- simplification of proofs as evaluation of programs

Example:

$$\dfrac{\dfrac{[A \wedge B]_1}{A}\,\wedge E_L}{(A \wedge B) \rightarrow A}\,\rightarrow I^1 \qquad \dfrac{\dfrac{x : [A \wedge B]_1}{fst(x) : A}}{\lambda x.fst(x) : (A \wedge B) \rightarrow A}$$

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Preliminary Notes

Categorial proof theory

- see [Došen, 1996], [Došen, 2001], [Lambek, 1974], [Lambek and Scott, 1986]
- Curry-Howard-Lambek correspondence
- objects interpreted as types/propositions and arrows as terms/proofs
- $f : A \vdash B$ as a code for a deduction that starts with premise A and ends with conclusion B

Preliminary Notes

Composition of deductions

- categorial proof theory: composition of deductions = composition of arrows

$$\frac{f : A \to B \qquad g : B \to C}{g \circ f : A \to C} \text{ ArrComp}$$

- proposition as types: composition of deductions = substitution

$$\frac{\Gamma \vdash a : A \qquad x : A, \Delta \vdash b : B}{\Gamma, \Delta \vdash b[a/x] : B} \text{ subs-ND}$$

- Example:

$$\frac{\Gamma \vdash A \wedge B \qquad A \wedge B \vdash B}{\Gamma \vdash B}$$

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Preliminary Notes

Associativity of deductions

- Associativity of deduction composition = permutation of cut

> ... the binary operation of composition [...] which in terms of deductions is a simple form of cut of sequent systems... [Došen, 2016]

$$\dfrac{\dfrac{f : A \to B \quad g : B \to C}{g \circ f : A \to C} \quad h : C \to D}{h \circ (g \circ f) : A \to D} \qquad \dfrac{f : A \to B \quad \dfrac{g : B \to C \quad h : C \to D}{h \circ g : B \to D}}{(h \circ g) \circ f : A \to D}$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

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Composition of Deductions

Deductions 1/4

- Došen: "Curry-Howard correspondence makes prominent proofs, while categorial proof theory is about deductions"
- systems built around the propositions as types principle, such as, e.g., constructive type theory ([Martin-Löf, 1984], CTT), are about deductions as well, they just have a different name for them: *hypothetical judgments*

Composition of Deductions

Deductions: 2/4

- In CTT, we start with categorical judgments

$$a : A$$

and generalize them into hypothetical judgments

$$x : A \vdash b(x) : B(x)$$

i.e., judgments depending on some assumptions, while the meaning of the latter is explained w.r.t. the former

- However, that does not mean that hypothetical notions are dispensible in CTT

Composition of Deductions

Deductions: 2/4

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Composition of Deductions

Deductions: 3/4

Consider, e.g., the rule for implication introduction:

$$\dfrac{\begin{array}{c} x : A \\ b(x) : B \end{array}}{\lambda x.b(x) : A \to B} \to\text{-intro}$$

where the *deduction* premise:

$$\begin{array}{c} x : A \\ b(x) : B \end{array}$$

is nothing other than a *hypothetical judgment*, i.e., a judgment with a context, that can be also written as $x : A \vdash b(x) : B$

Composition of Deductions

Deductions: 4/4

The relationship between $\vdash$ and $\rightarrow$, when A and B are considered as propositions, can be schematized as follows:

$$\underbrace{x : A \vdash b(x) : B}_{\text{hypothetical judgment, sequent, deduction}} \quad \Rightarrow \quad \underbrace{\lambda x.b(x) : A \rightarrow B}_{\text{categorical judgment, formula, proof}}$$

"deduction theorem"

- structural vs. logical information
- when considering a rule for composing deductions, we should think of hypothetical judgments and not of categorical ones

Composition of Deductions

Deductions: 4/4

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- when considering a rule for composing deductions, we should think of hypothetical judgments and not of categorical ones

Composition of Deductions

Composing deductions: 1/8

$$\frac{\dfrac{A \wedge B \vdash A \wedge B \qquad A \wedge B \vdash A}{A \wedge B \vdash A} \qquad A \vdash A \vee B}{A \wedge B \vdash A \vee B} \tag{1}$$

$$\frac{A \wedge B \vdash A \wedge B \qquad \dfrac{A \wedge B \vdash A \qquad A \vdash A \vee B}{A \wedge B \vdash A \vee B}}{A \wedge B \vdash A \vee B} \tag{2}$$

Composition of Deductions

Composing deductions: 2/8

$$\dfrac{\dfrac{c : A \wedge B \vdash c : A \wedge B \qquad x : A \wedge B \vdash \mathit{fst}(x) : A}{c : A \wedge B \vdash \mathit{fst}(c) : A} \qquad d : A \vdash \mathit{inl}(d) : A \vee B}{c : A \wedge B \vdash \mathit{inl}(\mathit{fst}(c)) : A \vee B}$$

$$\dfrac{c : A \wedge B \vdash c : A \wedge B \qquad \dfrac{x : A \wedge B \vdash \mathit{fst}(x) : A \qquad d : A \vdash \mathit{inl}(d) : A \vee B}{c : A \wedge B \vdash \mathit{inl}(\mathit{fst}(x)) : A \vee B}}{c : A \wedge B \vdash \mathit{inl}(\mathit{fst}(c)) : A \vee B}$$

Clearly $\mathit{inl}(\mathit{fst}(c)) = \mathit{inl}(\mathit{fst}(c)) : A \vee B$, but no associativity

Composition of Deductions

Composing deductions: 2/8

$$\dfrac{\dfrac{c : A \wedge B \vdash c : A \wedge B \qquad x : A \wedge B \vdash fst(x) : A}{c : A \wedge B \vdash fst(c) : A} \qquad d : A \vdash inl(d) : A \vee B}{c : A \wedge B \vdash inl(fst(c)) : A \vee B}$$

$$\dfrac{c : A \wedge B \vdash c : A \wedge B \qquad \dfrac{x : A \wedge B \vdash fst(x) : A \qquad d : A \vdash inl(d) : A \vee B}{c : A \wedge B \vdash inl(fst(x)) : A \vee B}}{c : A \wedge B \vdash inl(fst(c)) : A \vee B}$$

Clearly $inl(fst(c)) = inl(fst(c)) : A \vee B$, but no associativity

Composition of Deductions

Composing deductions: 3/8

Composition of arrows in category theory corresponds to substitution in constructive type theory: the arrows are interpreted as terms, objects as types:

$$\frac{x : A \vdash b(x) : B \qquad y : B \vdash c(y) : C}{x : A \vdash c(b(x)) : C}\ \text{CompDed}$$

We can rewrite $c(b(x))$ as $(c \circ b)(x)$

Composition of Deductions

Composing deductions: 4/8

$$\dfrac{\dfrac{x : A \vdash b(x) : B \qquad y : B \vdash c(y) : C}{x : A \vdash (c \circ b)(x) : C} \qquad z : C \vdash d(z) : D}{x : A \vdash (d \circ (c \circ b))(x) : D}$$

$$\dfrac{x : A \vdash b(x) : B \qquad \dfrac{y : B \vdash c(y) : C \qquad z : C \vdash d(z) : D}{y : B \vdash (c \circ b)(y) : D}}{x : A \vdash ((d \circ c) \circ b)(x) : D}$$

Composition of Deductions

Composing deductions: 5/8

- There is, however, a problem with the `CompDed` rule as presented
- the y in $c(y)$ in the second premise of the `CompDed` rule has to be free, otherwise the compositionality breaks down
- yet we cannot generally guarantee that c contains a free variable
- we need to find a more general way to represent deductions of the general form “from A can be deduced B”

Composition of Deductions

Composing deductions: 6/8

- we can achieve this with the higher-order presentation of CTT (see, e.g., [Nordström et al., 2001]) by using the notion of functional abstraction
- it allows us to express and generalize the functional content of hypothetical judgments (deductions) such as $x : A \vdash b : B$
- assuming A and B are types, we can form a new type $(A)B$, which can be populated by the following rule for functional abstraction:

$$\frac{x : A \vdash b : B}{(x)b : (A)B}$$

Composition of Deductions

Composing deductions: 7/8

- deductions can be treated as objects of higher-order function types. Changing the rule CompDed accordingly, we get:

$$\frac{f : (A)B \qquad g : (B)C}{(g \circ f) : (A)C}\ \text{CompDed*}$$

where $(g \circ f)(x) : C$ is defined in a standard manner as $g(f(x)) : C$ in the context $x : A$.

Composition of Deductions

Composing deductions: 8/8

$$\cfrac{\cfrac{f : (A \wedge B) A \wedge B \qquad \mathbf{fst} : (A \wedge B) A}{\mathbf{fst} \circ f : (A \wedge B) A} \qquad \mathbf{inl} : (A) A \vee B}{\mathbf{inl} \circ (\mathbf{fst} \circ f) : (A \wedge B) A \vee B} \tag{3}$$

$$\cfrac{f : (A \wedge B) A \wedge B \qquad \cfrac{\mathbf{fst} : (A \wedge B) A \qquad \mathbf{inl} : (A) A \vee B}{\mathbf{inl} \circ \mathbf{fst} : (A \wedge B) A \vee B}}{(\mathbf{inl} \circ \mathbf{fst}) \circ f : (A \wedge B) A \vee B} \tag{4}$$

For different permutations of cut we have different yet equivalent proof objects: $(\mathbf{inl} \circ (\mathbf{fst} \circ f)) = ((\mathbf{inl} \circ \mathbf{fst}) \circ f)$.

Composition of Deductions

Composing deductions: 8/8

$$\dfrac{\dfrac{f : (A \wedge B)A \wedge B \qquad \mathbf{fst} : (A \wedge B)A}{\mathbf{fst} \circ f : (A \wedge B)A} \qquad \mathbf{inl} : (A)A \vee B}{\mathbf{inl} \circ (\mathbf{fst} \circ f) : (A \wedge B)A \vee B} \tag{3}$$

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Conclusion

- contrary to Došen's claims, the propositions-as-types paradigm does not favour categorical proofs over inferences
- associativity of deduction composition does not have to become invisible
- we have demonstrated this in CTT, where deductions are understood in terms of hypothetical judgments
- from these hypothetical judgments we can derive higher-order judgments that we can compose and keep track of their associativity

Conclusion

Thank you for your attention.

References I

[Curry and Feys, 1958] Curry, H. B. and Feys, R. (1958).
Combinatory Logic.
Combinatory Logic. North-Holland, Amsterdam.

[De Bruijn, 1968] De Bruijn, N. (1968).
{Automath}, a language for mathematics.
Technical report, Department of Mathematics, Eindhoven University of Technology.

[Došen, 1996] Došen, K. (1996).
Deductive Completeness.
Bulletin of Symbolic Logic, 2(03):243–283.

[Došen, 2001] Došen, K. (2001).
Abstraction and application in adjunction.
In Kadelburg, Z., editor, *Proceedings of the Tenth Congress of Yugoslav Mathematicians, Faculty of Mathematics*, pages 33–46, Belgrade. University of Belgrade.

[Došen, 2015] Došen, K. (2015).
Inferential Semantics.
In Wansing, H., editor, *Dag Prawitz on Proofs and Meaning*, pages 147–162. Springer International Publishing, Switzerland.

[Došen, 2016] Došen, K. (2016).
On the Paths of Categories.
In Piecha, T. and Schroeder-Heister, P., editors, *Advances in Proof-Theoretic Semantics*, pages 65–77. Springer International Publishing, Cham.

Conclusion

References II

[Howard, 1980] Howard, W. A. (1980).
The formulae-as-types notion of construction.
In Curry, H. B., Hindley, J. R., and Seldin, J. P., editors, *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus, and Formalism*. Academic Press, London.

[Kreisel, 1971] Kreisel, G. (1971).
A Survey of Proof Theory II.
In Renstad, J. E., editor, *Proceedings of the Second Scandinavian Logic Symposium,*, page 109–170, Amsterdam. North-Holland.

[Lambek, 1974] Lambek, J. (1974).
Functional completeness of cartesian categories.
Annals of Mathematical Logic, 6(3-4):259–292.

[Lambek and Scott, 1986] Lambek, J. and Scott, P. J. (1986).
Introduction to higher order categorical logic.
Cambridge University Press.

[Martin-Löf, 1984] Martin-Löf, P. (1984).
Intuitionistic type theory: Notes by Giovanni Sambin of a series of lectures given in Padua, June 1980.
Bibliopolis, Napoli.

[Nordström et al., 2001] Nordström, B., Petersson, K., and Smith, J. M. (2001).
Martin-Löf's type theory, Handbook of logic in computer science: Volume 5: Logic and algebraic methods.
Oxford University Press, Oxford.

[Pezlar, 2020] Pezlar, I. (2020).
Composition of Deductions within the Propositions as Types Paradigm.
Logica Universalis, online(14):481–493.

Conclusion

References III

[Prawitz, 1972] Prawitz, D. (1972).
The Philosophical Position of Proof Theory.
In Olson, R. E. and Paul, A. M., editors, *Contemporary Philosophy in Scandinavia*, page 123–134. John Hopkins Press, Baltimore.