Bisimulations between Veltman models and generalized Veltman models

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- **ILM**: $A \rhd B \to (A \wedge \Box C) \rhd (B \wedge \Box C)$ (Montagna's principle)

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Satisfaction: $w \Vdash A \rhd B$ if for all u s.t. wRu and $u \Vdash A$ there is v s.t. $uS_w v$ and $v \Vdash B$

Generalized semantics

Generalized Veltman models:

- $W \neq \emptyset$
- $R \subseteq W \times W$ transitive and reverse well-founded
- for each $w \in W$, $S_w \subseteq R[w] \times \mathcal{P}(R[w])$
  - if wRu then $uS_w\{u\}$
  - if uS_wV and vS_wZ_v for all $v \in V$ then $uS_w(\cup Z_v)$
  - if $wRuRv$ then $uS_w\{v\}$

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Bisimulation between Veltman models

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Key properties:

- if wZw', then w and w' are modally equivalent
- the converse does not hold generally, but it holds in case of image-finite Veltman models (an analogue of Hennessy-Milner theorem, de Jonge 2004)

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- Hennessy-Milner analogue does not hold

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- Hennessy-Milner analogue holds

Example

Consider a generalized Veltman frame such that:

- $W = \{0, 1, 2, 3\}$, $R = \{(0, 1), (0, 2), (0, 3)\}$, $1S_0\{2, 3\}$
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Then $Z = \{(0, 0'), (1, 1'), (1, 1''), (2, 2'), (3, 3')\}$ is a bisimulation. Hence, 0 and $0'$ are modally equivalent (as are all pairs in Z). With the more restrictive definition of bisimulation, we would not have a bisimulation in this example, thus we can use it as a counterexample for Hennessy-Milner analogue in that case.

Obtaining a bisimilar model

It is straightforward to obtain a bisimilar generalized Veltman model from a given Veltman model: we use the same W and R, and define $uS'_w V$ iff $uS_w v$ for some $v \in V$.

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The previous example is very simple, but already illustrates that the opposite direction is much more involved. Exploring it is an ongoing work.