Structures for relative quantifier elimination in valued fields

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Valued fields

A *valued field* is a field K with a *valuation* v, which is a map from K onto $vK \cup \{\infty\}$, where $(vK, +, 0, \leq)$ is an ordered abelian group and

- $v(x) = \infty$ if and only if $x = 0$,
- $v(xy) = v(x) + v(y)$,
- $v(x + y) \geq \min\{v(x), v(y)\}$.

The ordered abelian group vK is called the *value group* of the valued field (K, v).

Language of valued fields

Let (K, v) be a valued field. The *valuation ring* of v

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The *language of valued fields* is the language of fields with a unary relation symbol $\mathcal{O}$ (to be interpreted as the valuation ring).

Residue field

The valuation ring of a valued field (K, v) has a unique maximal ideal

$$\mathcal{M}_v := \{x \in K \mid vx > 0\}.$$

The quotient $Kv := \mathcal{O}_v/\mathcal{M}_v$ is called the *residue field* of (K, v). The characteristic of Kv is called the *residue characteristic* of (K, v).
There are only three possibilities for the pair $(\operatorname{char} K, \operatorname{char} Kv)$, namely $(0, 0)$, $(0, p)$ or (p, p) where $p > 0$ is a prime number.
We will consider the first two cases for *henselian* valued fields.

Example

Let p be a prime number. We define the p-adic valuation v_p on $\mathbb{Q}$. For $x \in \mathbb{Q}^\times$ we write

$$x = p^{v_p(x)} \frac{a}{b}$$

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As a complete valued field of rank 1, it is henselian. It is the prototype of p-adically closed fields as $\mathbb{R}$ is the prototype of real closed fields.

Some history

- MacIntyre (1976) shows quantifier elimination for the theory of p-adically closed fields in the language of valued fields extended with power predicates:

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- Pas (1989) shows quantifier elimination for the theory of henselian valued fields of residue characteristic 0, relative to the value group and the residue field, in the Denef-Pas language with angular component map.

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- In 1994, Kuhlmann simplifies the structures of Basarab introducing the structures of additive and multiplicative congruences (amc-structures).
- In 2011, Flenner simplifies the structures introduced by Kuhlmann even further, with the leading term structures (RV-structures).

RV-structures

Definition (Flenner)

Let (K, v) be a valued field and $\gamma \in vK_{\geq 0}$. The *RV-structure of level γ of* (K, v) is

$$K^{\times}/1 + \mathcal{M}^{\gamma} \cup \{\mathbf{0}\}$$

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with its multiplicative structure and a ternary relation

$$\begin{aligned} &\oplus_{\gamma}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \iff \\ &\quad \exists x, y, z \in K : rv_{\gamma}(x) = \mathbf{x} \wedge rv_{\gamma}(y) = \mathbf{y} \\ &\quad \wedge rv_{\gamma}(z) = \mathbf{z} \wedge x + y = z \end{aligned}$$

Multivalued operations

A *hyperoperation* (or *multivalued operation*) on a set $F \neq \emptyset$ is a function $+$ which associates to every pair $(x, y) \in F \times F$ a non-empty subset of F, denoted by $x + y$.

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If $+$ is a hyperoperation on F and $A, B \subseteq F$, then we set

$$A + B := \bigcup_{a \in A, b \in B} a + b.$$

If $x \in F$ we abbreviate $A + \{x\} =: A + x$.

Canonical hypergroups

A *canonical hypergroup* is a tuple $(F, +, 0)$, where $+$ is a hyperoperation on F and 0 is an element of F such that the following axioms hold:

- the hyperoperation $+$ is associative, i.e., $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$,
- $x + y = y + x$ for all $x, y \in F$,
- for every $x \in F$ there exists a unique $x' \in F$ such that $0 \in x + x'$ (the element x' is denoted by $-x$),
- $z \in x + y$ implies $y \in z - x := z + (-x)$ for all $x, y, z \in F$.

Krasner's Hyperrings and hyperfields

A *(commutative) hyperring* is a tuple $(F, +, \cdot, 0)$ which satisfies the following axioms:

- $(F, +, 0)$ is a canonical hypergroup,
- $(F, \cdot)$ is a commutative semigroup and 0 is an absorbing element, i.e., $x \cdot 0 = 0$ for all $x \in F$,
- the operation $\cdot$ is distributive over the hyperoperation $+$. That is, for all $x, y, z \in F$,

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

A *hyperfield* is a hyperring such that $F^{\times} := F \setminus \{0\}$ is a group under multiplication.

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Given a field K and a subgroup T of $K^\times$, one can always construct a hyperfield, called the *factor hyperfield* of K modulo T, denoted by K_T. As a set it is $K^\times/T \cup \{[0]_T\}$. The hyperoperation is defined as follows:

$$[x]_T + [y]_T := \{[x + yt]_T \in K_T \mid t \in T\},$$

where for $x \in K^\times$, the coset xT is denoted by $[x]_T$.

The γ-valued hyperfields

Definition

Let (K, v) be a valued field and $\gamma \in vK_{\geq 0}$. The *γ-valued hyperfield* of (K, v) is the factor hyperfield $K_{1+\mathcal{M}^\gamma}$ also denoted by $\mathcal{H}_\gamma(K)$.

$\mathcal{H}_\gamma(K)$ is a *valued hyperfield* since $1 + \mathcal{M}^\gamma \subseteq \mathcal{O}_v^\times = \ker v$.

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$\mathcal{H}_\gamma(K)$ is a *valued hyperfield* since $1 + \mathcal{M}^\gamma \subseteq \mathcal{O}_v^\times = \ker v$.
A hyperoperation $+$ can be encoded by a ternary relation symbol:

$$r_+(x, y, z) \iff z \in x + y.$$

Valued hyperfields and RV-structures

Let (K, v) be a valued field and $\gamma \in vK_{\geq 0}$.
As sets, $\mathcal{H}_\gamma(K)$ and the RV-structure of level γ of (K, v) are the same thing.
The relation which encodes the hyperoperation of $\mathcal{H}_\gamma(K)$ is the same as Flenner's relation $\oplus_\gamma$.

Valued hyperfields and graded rings

Definition

Let (K, v) be a valued field. The *graded ring associated to* (K, v) is

$$\mathrm{gr}_v(K) := \bigoplus_{\gamma \in vK} \mathcal{P}^\gamma / \mathcal{M}^\gamma,$$

where $\mathcal{P}^\gamma := \{x \in K \mid vx \geq \gamma\}$.

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There is a language $\mathcal{L}_{gr}$ extending the language of rings such that $\mathrm{gr}_v(K)$ is an $\mathcal{L}_{gr}$-structure and the hyperfield structure of $\mathcal{H}_0(K)$ is interpretable in $\mathrm{gr}_v(K)$.

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Theorem

Let (L,w) and (F,u) be henselian valued fields of residue characteristic 0 and (K,v) a common valued subfield. The following are equivalent:

- $(L,w) \equiv_{(K,v)} (F,u)$;

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- $\mathrm{gr}_w(L) \equiv_{\mathrm{gr}_v(K)} \mathrm{gr}_u(F)$ *as $\mathcal{L}_{gr}$-structures.*

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Theorem

Let (L, w) and (F, u) be henselian valued fields of characteristic 0 and residue characteristic $p > 0$. Let (K, v) be a common valued subfield. The following are equivalent:

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- $\mathcal{H}_{n \cdot vp}(L) \equiv_{\mathcal{H}_{n \cdot vp}(K)} \mathcal{H}_{n \cdot vp}(F)$ *for all* $n \in \mathbb{N}$;

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Remark

The graded rings are not sufficient in the mixed characteristic case.

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