de Finetti coherence in infinitary logic

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Logic 4 Peace

1. Probability in Łukasiewicz logic
2. A well behaved infinitary logic
3. A Dutch-book theorem
4. An application to statistical models

MV-algebras

- Algebraic counterpart of Łukasiewicz logic,
- Unit interval of lattice ordered groups with strong unit,
- A generalization of boolean algebras.

For $x, y \in [0,1]$ define

$$x \oplus y = \min(x+y, 1) \quad \text{and} \quad \neg x = 1 - x, \quad 0, 1$$

A is an MV-algebra iff $A \in HSP([0,1])$

Probability on MV-algebras: states

A an MV-algebra, $s : A \to [0,1]$ such that

- $s(1) = 1$,
- $s(x \oplus y) = s(x) + s(y)$ when $x \odot y = 0$.

Mundici D., *Averaging the Truth-Value in Łukasiewicz Logic*, Studia Logica 55 (1995) 113-127.

$$\mathcal{S}(A) = \{s : A \to [0,1] \mid s \text{ is a state of } A\}$$

Important properties:

States of A form a compact convex subset of $[0,1]^A$

$\mathcal{S}(A)$ is the topological closure of $conv(Hom(A,[0,1]))$, where the topology is the product topology on $[0,1]^A$.

If $Max(A)$ is the set of maximal ideal of A, topologized with the usual hull-kernel topology,

there is an affine homeomorphism (that is, it preserves convex combinations) between $\mathcal{S}(A)$ and $\mathcal{M}(Max(A))$, where $\mathcal{M}(Max(A))$ is the space of Borel probability measures on $Max(A)$.

On algebras of functions,

$$s(f) = \int_{Max(A)} f d\mu,$$

de Finetti's foundation of probability

De Finetti's theory of subjective probability is based on the notion of coherence:

two players, bookmaker (**B**) and gambler (**G**), wage on a certain class of events of interest.

A system of bets is coherent if there is no way for **B** to incur in a sure loss, i.e.

if there is no way for **G** to arrange her stakes in order to win money independently of the result of the events involved in the bet.

Take an integer $n > 0$, let $\varphi_1, \dots, \varphi_n$ formulas in the language of classical logic. These are our events of interest.

B publishes a book $\mathrm{b} = \{(\varphi_i, \beta_i) \mid i = 1, \dots, n\}$ assigning to each event a value in $[0, 1]$.

G chooses the stakes $\sigma_1, \dots, \sigma_n \in \mathbb{R}$, and pays $\sum_{i=1}^{n} \sigma_i \beta_i$ to B.

Let now v be a possible state of world: formally, we can think of it as a boolean evaluation on the formulas $\varphi_1, \dots, \varphi_n$. Then, in v, for any φ_i, B pays G

- 0 if φ_i is false in v
- σ_i if φ_i is true in v

The total balance of this bet in the possible world v is

$$\sum_{i=1}^{n} \sigma_i \beta_i - \sum_{i=1}^{n} \sigma_i v(\varphi_i) = \sum_{i=1}^{n} \sigma_i (\beta_i - v(\varphi_i)).$$

Many-valued coherence criterion

Replace classical logic with Łukasiewicz logic and evaluations in $\{0, 1\}$ with evaluations (=homomorphisms) in $[0, 1]$...

Let A be an MV-algebra, $E = \{e_1, \ldots, e_n\} \subseteq A$ the set of events.
$\beta \in [0, 1]^n$, a book on E, is said to be coherent if for any choice of stakes $\sigma_1, \ldots \sigma_n \in \mathbb{R}$ by G, there exists $h \in Hom(A, [0, 1])$ such that

$$\sum_{i=1}^{n} \sigma_i(\beta_i - h(e_i)) \geq 0$$

The no-Dutch book theorem

Theorem

Let A an MV-algebra, $E = \{e_1, \ldots, e_n\}$ a finite set of events and $\beta \in [0,1]^n$ a book on E. TFAE:

1. *β is coherent.*
2. *β can be extended to a convex combination of points at most $n-1$ points in $hom(A, [0,1])$.*
3. *there exists a state $s \in \mathcal{S}(A)$ such that $\beta_i = s(e_i)$.*

Main point of the proof:

$$\mathcal{S}(A) = \overline{conv}(Hom(A, [0,1]))$$

What about infinitary logic?

The main ingredient: RMV_σ

MV-algebras closed under scalar operation and countable suprema, endowed with σ-homomorphisms of Riesz MV-algebras.

We can think of algebras in $\mathbf{RMV}_\sigma$ as infinitary algebras: the countable suprema $\bigvee$ is an operation of countable arity in the language

$$\oplus, \neg, 0, \bigvee, \{\alpha\}_{\alpha \in [0,1]}.$$

$\mathrm{RMV}_\sigma = \mathit{HSP}([0,1])$

It is an infinitary variety of algebras! [Di Nola, Lapenta, Leuştean, 2018]

We defined a logical systems that it is standard complete wrt $[0,1]$ and has, as intented semantics, $\mathbf{RMV}_\sigma$.

Free algebras for countably-many generators in RMV_σ

Theorem (Di Nola, L., Lenzi, 2021 — Di Nola, L., Leuştean, 2018)

The free κ-generated algebra in RMV_σ is the algebra

$$Borel([0,1]^\kappa) = \{a : [0,1]^\kappa \to [0,1] \mid a \text{ is Borel-measurable}\},$$

generated by the projections π_i, $i \in \kappa$ and $\kappa \leq \omega$.

A privileged subclass

An algebra $A \in \mathrm{RMV}_\sigma$ is σ-semisimple if, and only if, $A \in ISP([0,1])$. Equivalently, there exists $\kappa \leq \omega$ and $V \in \mathcal{BO}([0,1]^\kappa)$ such that

$$A \simeq \mathrm{Borel}([0,1]^\kappa)|_V$$

Conditional events and coherence

A conditional event in a σ-semisimple algebra A is a pair $(p, q) \in A \times A$.

Given conditional and unconditional events in A, a conditional book is the assignment

$$\beta\colon (p_1, q_1) \mapsto \alpha_1, \ldots, (p_n, q_n) \mapsto \alpha_n, r_1 \mapsto c_1, \ldots, r_m \mapsto c_m$$

where $\alpha_i, c_j \in [0, 1]$.

A book is said complete if for any q_i there exists a unique index j such that $q_i = r_j$, that is $\{q_1, \ldots q_n\} \subseteq \{r_1, \ldots, r_m\}$ with no repetitions.

Conditional events and coherence

If $A \simeq \text{Borel}([0,1]^\kappa)|_V$, the book β is conditionally coherent if, and only if, for any $\sigma_1, \ldots, \sigma_n \in \mathbb{R}$ and $\delta_1, \ldots, \delta_m \in \mathbb{R}$ there exists $\mathrm{x} \in V$ such that

$$\sum_{t=1}^{n} \sigma_t q_t(\mathrm{x})(\alpha_t - p_t(\mathrm{x})) + \sum_{j=1}^{m} \delta_j (c_j - r_j(\mathrm{x})) \geq 0$$

The main result

Let V be a G_δ subset of $[0,1]^\kappa$ and $A \simeq \mathrm{Borel}([0,1]^\kappa)|_V$.

Theorem

The complete book

$$\beta\colon (p_1, q_1) \mapsto \alpha_1, \ldots, (p_n, q_n) \mapsto \alpha_n, r_1 \mapsto c_1, \ldots, r_m \mapsto c_m$$

is conditionally coherent if, and only if, there exists a σ-state s on $A \simeq \mathrm{Borel}([0,1]^\kappa)|_V$ such that $s(p_i \cdot q_i) = \alpha_i s(q_i)$ and $s(r_j) = c_j$, for the obvious choices of the indexes.

Steps of the proof

- Reduce to the case of unconditional books
- Prove the analogous of de Finetti's theorem for MV-algebras$\rightarrow$ the same proof strategy does not work!

An application: Logico-algebraic statistical models

$$\eta = (\eta_i)_{i \leq \kappa} \colon P \to \Delta_\kappa,$$

$P \subseteq \mathcal{BO}([0,1]^d)$ and Δ_κ is the standard κ-dimensional simplex.

- $P \subseteq [0,1]^d$ is the set of states of the world, or parameters;
- $\eta = (\eta_i)_{i \leq \kappa} \colon P \to [0,1]^k$ is our statistical model: to each parameter $\mathrm{x} \in P$ it associates the tuple $(\eta_i(\mathrm{x}))_{i \leq \kappa}$.

Coherent model

The model $\eta : P \subseteq [0,1]^d \to [0,1]^k$ is coherent with respect to the events $E = \{p_1, \ldots, p_k\} \subseteq \mathrm{Borel}([0,1]^n)$ iff for any $\mathrm{x} \in P$ there exists a state $s \colon \mathrm{Borel}([0,1]^n) \to [0,1]$ such that such that $s(p_i) = \eta_i(\mathrm{x})$ for any $i = 1, \ldots, k$

An example

Let $k \in \mathbb{N}$ and let us consider a binomial model:

$$\eta = (\eta_0, \ldots, \eta_k) \colon [0,1] \to [0,1]^{k+1},$$

$$\eta_i \colon [0,1] \to [0,1] \quad \eta_i(x) = \binom{k}{i} x^i (1-x)^{k-i}.$$

k represents the iterations of an experiment,
$\eta_i(x)$ is the probability of having i successes and $k-i$ failures, given that the probability of success in one single trial is x.

Since $\sum_{i=0}^{k} \eta_i(x) = 1$, such a model is always coherent with respect to any set $\{p_1, \ldots, p_k\}$ that satisfies the following conditions:
(1) $\bigoplus_{i \neq j} p_i = \neg p_j$
(2) for any i, there exists $x \in [0,1]$ such that $p_i(x) = 1$.

Thank you!