Modal μ-calculus and alternating parity tree automata: a direct translation

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Introduction

About the modal μ-calculus :

- useful for model checking
- contains temporal logics LTL, CTL. . .
- equivalent to monadic second order logic over trees
- equivalent automata models allow to reason more easily on its semantics using the parity condition (the one from parity games)

Bibliography

- [EJ91] An article from E. Allen Emerson and Charanjit S. Jutla which gives the translation from μ-calculus to tree automata only for the Streett condition which is equivalent to the parity one. We wanted parity condition and there is no converse translation.
- [JW95] An article from Janin and Walukiewicz which gives a complete translation but only for disjunctive formulas. They prove it is always possible to put a formula in disjunctive form, so there is an equivalence but the translation is not direct and the proof is quite short, we wanted more details.
- [Wil01] An article from Thomas Wilke which translates μ-calculus to automata, but not the converse. The automata formalism is quite different than ours.

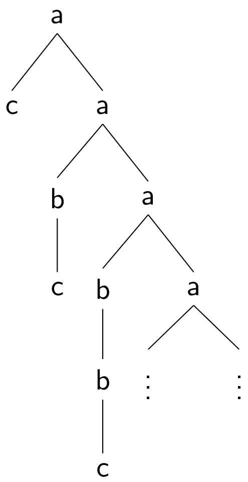
Objectives of our direct translation

We do not know any direct "bidirectional" translation between μ-calculus formulas and the alternating parity tree automata we will present later on.

Our aim is to use this direct translation to rewrite the Kobayashi-Ong type system [KO09] in a way that will reveal a connection with cyclic proofs.

Tree and signature

We consider Σ-labelled ranked tree defined with a signature. Here is an example defined over the signature $\Sigma = \{a : 2, b : 1, c : 0\}$:



Modal μ-calculus : syntax

Consider a signature $\Sigma = \{a : ar(a), b : ar(b), c : ar(c)...\}$ and a set of variable $Var = \{X, Y, Z....\}$.

The grammar is defined inductively :

$$\varphi \ ::= \ X \,|\, \underline{a} \,|\, \varphi \vee \varphi \,|\, \varphi \wedge \varphi \,|\, \Box\varphi \,|\, \diamond_i \varphi \,|\, \diamond \varphi \,|\, \mu X.\varphi \,|\, \nu X.\varphi$$

Example of formulas :

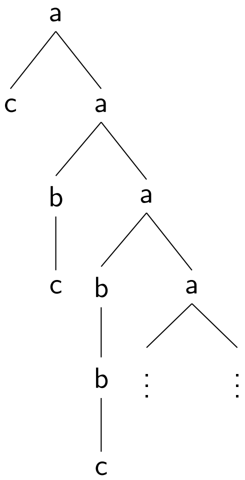
- $\mu X. \diamond X$
- $\nu X.(\Box X \wedge \mu Y.(a \vee \diamond Y \vee \nu Z.(b \wedge \Box Z)))$

Modal μ-calculus : semantics

Consider a valuation function $V : Var \longrightarrow \mathcal{P}(N)$, then :

- $||a||_V = \{n \in N \mid n \text{ is labelled by } a\}$
- $||X||_V = V(X)$
- $||\neg\varphi||_V = N \backslash ||\varphi||_V$
- $||\varphi \vee \psi||_V = ||\varphi||_V \cup ||\psi||_V$
- $||\varphi \wedge \psi||_V = ||\varphi||_V \cap ||\psi||_V$
- $|| \diamond_i \varphi||_V = \{n \in N \mid \mathrm{ar}(n) \geq i \text{ and } succ_i(n) \in ||\varphi||_V\}$
- $||\mu X.\varphi(X)||_V = \bigcap\{M \subset N \mid ||\varphi||_{V[X \leftarrow M]} \subset M\}$
- $||\nu X.\varphi(X)||_V = \bigcup\{M \subset N \mid M \subset ||\varphi||_{V[X \leftarrow M]}\}$

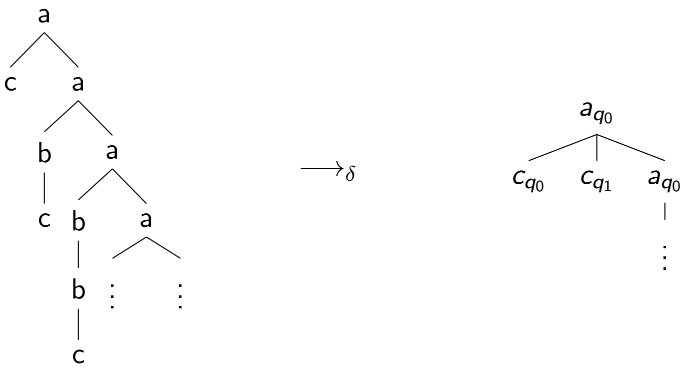
Example of semantics



$$\varphi_1 = ||\nu X.(\diamond X \vee c)||$$
$$\varphi_2 = ||\mu X.(\Box X \vee c)||$$

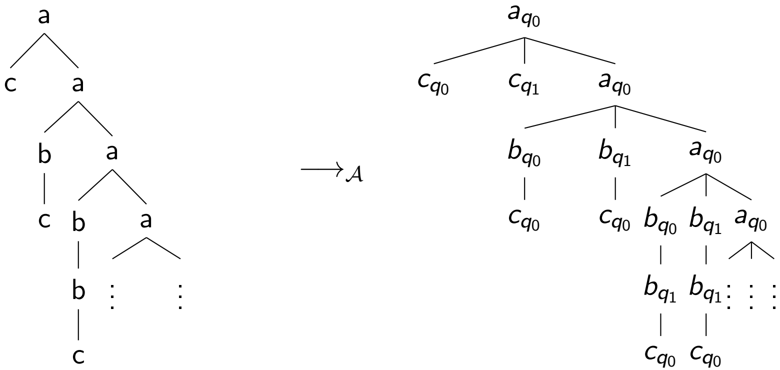
Alternating parity tree automata (APTA) over an example

We consider an APTA $\mathcal{A}$ with states q_0 and q_1, initial states q_0 :
$\delta(a, q_0) = (1, q_0) \wedge (1, q_1) \wedge (2, q_0)$:



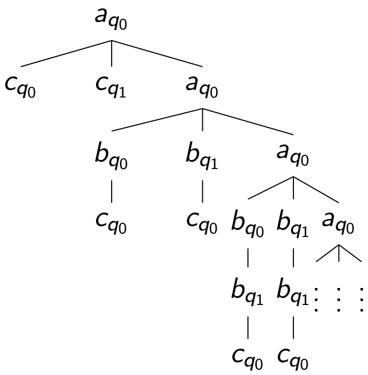
Alternating parity tree automata (APTA) over an example

If we add the following transitions : $\delta(b, q_0) = (1, q_0) \vee (1, q_1), \delta(b, q_1) = (1, q_0) \vee (1, q_1), \delta(c, q_0) = \top, \delta(c, q_1) = \top\}$ One possible transcription is :



Alternating parity tree automata (APTA) over an example

We use the coloring $\Omega(q_0) = 0$ and $\Omega(q_1) = 1$.



The infinite branch is accepted.

Translation from the modal μ-calculus to APTA

- We translate it by induction on the formula syntax.
- Every state of the resulting automaton correspond to a subformula.
- For the coloration of the state, we use the alternation depth.

Example

The following formula has alternation depth 2 :

$$\nu X.(\Box X \wedge \mu Y.(a \vee \diamond Y \vee \nu Z.(b \wedge \Box Z)))$$

The following formula has alternation depth 1 :

$$\nu X.(\Box X \vee \mu Y.(a \vee \diamond Y) \vee \nu Z.(b \wedge \diamond Z))$$

Translation from APTA to μ-calculus without parity condition

Principle : We put the transition functions in disjunctive form as for example $\delta(q_k, a_k) = \bigvee_{i \in I_k} \bigwedge_{j \in J_k} (d_{i,j}, q_{i,j})$ Each state correspond to a formula, the initial state is the resulting formula.

$$X_k = \bigvee_{a \in \Sigma} [a \wedge \bigvee_{i \in I_{a,k}} \bigwedge_{j \in J_{a,k}} \diamond_{d_{i,j}} X_{i,j}]$$

Example

We consider the signature $\Sigma = \{a : 2\}$ and the automaton $\mathcal{A} = \langle \Sigma, \mathcal{Q} = \{q_1, q_2\}, \delta = \{(q_1, a) : (1, q_2) \wedge (2, q_2), (q_2, a) : (1, q_1) \wedge (2, q_1)\}, q_1, \Omega = \{q_1 : 1, q_2 : 4\} \rangle$. We have :

$$X_1 = a \wedge \diamond_1 X_2 \wedge \diamond_2 X_2$$
$$X_2 = a \wedge \diamond_1 X_1 \wedge \diamond_2 X_1$$

Translation from APTA to μ-calculus : adding the parity condition

We add colored equalities with the same color that the state considered :

$$X_1 =_1 (a \wedge \diamond_1 X_2 \wedge \diamond_2 X_2)$$
$$X_2 =_4 (a \wedge \diamond_1 X_1 \wedge \diamond_2 X_1)$$

We say this system of equations is not **parity coherent** : there is a recursive call to the color 4 under the scope of a recursive call of color 1.

This is problematic to obtain the translation to modal μ-calculus formulas.

Obtaining parity coherence

We can use the rules to obtain a parity coherent term.

Set $X_1 = \rho_1 X_1.(a \wedge \diamond_1 X_2 \wedge \diamond_2 X_2) = \mathcal{R}(X_1)$.

Then X_1 rewrites to :

$$a \wedge \diamond_1(X_2[X_1 \leftarrow \mathcal{R}(X_1)]) \wedge \diamond_2(X_2[X_1 \leftarrow \mathcal{R}(X_1)])$$

emitting the color 1 (negligible from the point of view of the parity condition), and then this rewrites to :

$$a \wedge \diamond_1(\rho_4 X_2.(a \wedge \diamond_1 \mathcal{R}(X_1) \wedge \diamond_2 \mathcal{R}(X_1))) \wedge \diamond_2(\rho_4 X_2.(a \wedge \diamond_1 \mathcal{R}(X_1) \wedge \diamond_2 \mathcal{R}(X_1)))$$

which is parity coherent.

Conclusion

- We prove the equivalence between μ-calculus and APTA, and we give a direct translation from one structure to another.
- We use as an intermediate step the notion of parity coherence, that does not appear with other equivalent acceptance condition such as the Streett condition.

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