Towards pluralism: from Poincaré to Heyting – different kinds of intuition behind logic

Miriam Franchella – Dipartimento di Studi Storici -
Università degli studi di Milano-
Italy
miriam.franchella@unimi.it

Henri Poincaré

Poincaré was deeply convinced that mathematical proofs come out of "intuition".

In “La valeur de la science” (1900) Poincaré listed three types of intuition:

“first, the appeal to the senses and the imagination;

next generalization by induction, copied, so to speak, from the procedures of the experimental sciences;

finally, we have the intuition of pure number, whence arose the second of the axioms just enunciated, which is able to create the real mathematical reasoning” (Poincaré 2014, pp. 215- 216)

and divided mathematicians into two types: ‘Geometers’, that rely on the first type of intuition, ‘analysts’ that rely on the third.

Still, he added:

“The analysts, […] in order to be inventors, must, without the aid of the senses and imagination, have a **direct sense of what constitutes the unity of a piece of reasoning**, of what makes, so to speak, its soul and inmost life.” (Poincaré 2014, p.220)

In his “Science et méthode” (1908) he gave a psychological reconstruction of the mathematical process. He started from the fact that man has two selves, one unconscious and one conscious. Then he proceeded by metaphor. He described concepts as atoms which stand still when the mind is resting, attached to one of its ‘walls’. Then, during unconscious work, carried out on the basis of what had been reflected consciously, some atoms will detach themselves from the wall and thus be able to meet (and hook up in combination) with others set in motion or still hooked to the wall. “The mobilized atoms are therefore not any atoms whatsoever; they are those from which we might reasonably expect the desired solution. The mobilized atoms, are those from which the solution sought can reasonably be expected”

"All goes on as if the inventor were an examiner for the second degree who would only have to question the candidates who had passed a previous examination". (Poincaré 2014, p. 386-387).

The combinations whose beauty, elegance and harmony have the greatest impact on us, and are capable of arousing in us an intense aesthetic emotion, emerge into consciousness. The sensitivity to such emotions (typical of mathematicians) "once aroused, will call our attention to them, and thus give them occasion to become conscious". (Poincaré 2014, p. 392)

Beauty, elegance and harmony emerge from that specific combination, because **"the mind without effort can embrace their totality while realizing the details**". (Poincaré 2014, p. 391)

This is the core of Poincare's definitions of intuition: the ability to grasp the unity of a demonstration, its being ordered in a certain way, at a glance.

In the same work he stressed:
“A mathematical demonstration is not a simple juxtaposition of syllogisms, it is syllogisms *placed in a certain order,* and the order in which these elements are placed is much more important than the elements themselves. If I have the feeling, the **intuition**, so to speak, of this order, so as to **perceive at a glance the reasoning as a whole**, I need no longer fear lest I forget one of the elements, for each of them will take its allotted place in the array, and that without any effort of memory on my part […] this intuition of mathematical order, that makes us divine hidden harmonies and relations, can not be possessed by every one. (Poincaré 2014, p. 385)

The author affirmed that people can be divided into the following three categories: 1) those lacking in ‘this delicate and difficult to define sensibility’ and above average mnemonic and concentration strength – i.e. the majority; 2) those who possess this sensibility to a limited extent but have an extraordinary memory; 3) those whose sensibility is prodigious and accompanied by a non-significant memory advantage. The former are seen as incapable of understanding or creating mathematics, the second group can only understand it and the latter can also create it and will be its ‘inventors’.

Without intuition there be no invention, mathematical novelty, and without a (minimal) intuition one cannot understand the mathematical demonstrations of others. Therefore, even 'logicians', in the inventive stage, must appeal to intuition. "Pure logic could never lead us to anything but tautologies" (Poincaré 2014, p. 214). "This shows us that logic is not enough; that the science of demonstration is not all science and that intuition must retain its role as complement, I was about to say as counterpoise or as antidote of logic." (Poincaré 2014, p. 217)

Such intuitions, though very different, nevertheless see intuition as a unified vision. Namely, reasoning by recurrence (i.e. mathematical induction), 'contains, as it were, condensed into a single formula, an infinity of hypothetical syllogisms' arranged in cascade: the theorem is true for the number 1; if it is true for 1, then it is also true for 2; therefore it is true for number 2; if it is true for 2, it is also true for 3, and so on. The conclusion of each syllogism is the premise of the next syllogism. In mathematical induction, we simply lay down the minor premise of the first syllogism (i.e. 'the theorem holds for number 1') and the general formula that contains all the major ones as special cases (i.e. 'if the theorem holds for n-1, then it holds for n'). Thus a **unitary look at a sequence** that would be infinite is provided.

He specified further: “both the logicians and the intuitionalists (sic) have achieved great things that others could not have done. Who would venture to say whether he preferred that Weierstrass had never written or that there had never been a Riemann? (Poincaré 2014, p. 212)

Poincaré, who classified himself a ‘geometer’, expressed admiration for analysts who work without the aid of the imagination: “The majority of us, if we wished to see afar by pure intuition alone, would soon feel ourselves seized with vertigo”. (Poincaré 2014, p. 221).

He named two Germans with different mentalities and two Frenchmen with equally different mentalities. The Frenchmen were Bertrand and Hermite: "They were scholars of the same school at the same time; they had the same education, were under the same influences; and yet what a difference! [...] M. Bertrand is always in motion; now he seems in combat with some outside enemy, now he outlines with a gesture of the hand the figures he studies. Plainly he sees and he is eager to paint, this is why he calls gesture to his aid. With M. Hermite, it is just the opposite; his eyes seem to shun contact with the world; it is not without, it is within he seeks the vision of truth." (Poincaré, 2014, p. 211). For the Germans, he offered us the analytical Weierstrass ("you may turn through all his books without finding a figure" (Poincaré, 2014, p. 212)) and the geometer Riemann "each of his conceptions is an image that no one can forget, once he has caught its meaning" (ibid.).

Racist intuition

Poincaré did not link being an analyst/geometer to a race.
We feel the need to specify this fact due to a famous Klein's lecture, where a racial difference in own approach to mathematics was introduced.

Felix Klein in the VI Evanston lecture of 1893 (p. 42) had expressed the following views:

"Finally, it must be said that the degree of exactness of the intuition of space may be different in different individuals, perhaps even in different races. It would seem as if a strong naive space-intuition were an attribute pre-eminently of the Teutonic race, while the critical, purely logical sense is more fully developed in the Latin and Hebrew races. A full investigation of this subject, somewhat on the lines suggested by Francis Galton in his researches on heredity, might be interesting."

Notice that Klein did not blame anybody for his race. For example, he praised James Joseph Sylvester with numerous adjectives: 'Sylvester was extremely engaging, witty and effervescent. He was a brilliant orator and often distinguished himself by his pithy, agile poetic skill, to the mirth of everyone' and then considered Sylvester's best traits to be typical of his 'race': 'By his brilliance and agility of mind he was a genuine representative of his race; he hailed from a purely Jewish family, which, having been nameless before, had adopted the [sur]name Sylvester only in his generation.' (Klein, 1926, p. 163; transl. Rowe, 1986, p. 440).

He also expressed positive views of Jewish mathematician Kronecker:

"In that he was mainly concerned with arithmetic and algebra, in later years however setting up definite intellectual norms for all mathematical work, he appears as the specifically Jewish talent, but in a special, individual enhancement. For he has foreseen many relationships of a fundamental nature in his fields of work, without being able to work them out clearly yet." (Klein, 1926, p. 281, transl. Rowe, 1986, p. 442)

Klein reserved the same treatment for Jewish Jacobi:

"As is well known, the year 1812 brought with it the emancipation of the Jews in Prussia. Jacobi was the first Jewish mathematician to take a leading place in Germany, and in so doing he was again at the forefront of a great, and for our science significant, development. This measure opened up a large reservoir of new mathematical talent for our country, whose powers, along with those of the French immigrants, very soon bore fruit. It appears to me that our science has won a strong stimulant through this type of blood replenishment. Along with the already mentioned law regarding shifts of productivity from country to country, I would like to designate this phenomenon as the effect of national infiltration." (Klein, 1926, p. 114; transl. Rowe, 1986, p. 440)

However, this distinction was taken up by others, starting with Erich Rudolf Jaensch, a former student of his who stated that Klein was intrigued by the 'conflict between the German spirit and the preponderance of a completely different type of thinking in mathematics' and continually returned to this theme in his seminar 'despite the fact that it was intentionally repressed by several of the participants.' (Jaensch-Althoff 1939, p. 32; transl. Rowe, 1986, p. 440).

According to Rowe (1986, p. 441), it is doubtful that Jaensch ever attended this seminar!

Later, Theodor Vahlen, who was an executive official in the ministry in 1933, and a professor in Berlin, gave an address on assuming his office as rector of University of Greifswald on 15 May 1923 entitled "Wesen und Wert der Mathematik“, where he quoted Klein (‘one of our greatest geometers’) stating that ‘modern people have a strongly developed, fertile view of space, which is a particular advantage of the Teutonic Race’ and that a ‘purely logical, sharply critical sense’ characterizes the Jews, generating a ‘disintegrating criticalism’ (Vahlen, 1923, p. 21). Cited in this way, Klein’s distinctions within mathematics culminated in open Antisemitism.

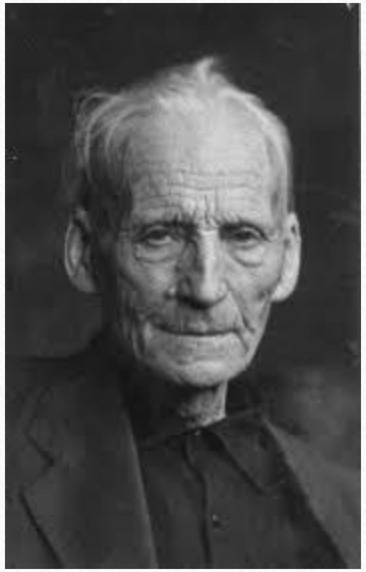
A quarter of a century after writing his thesis under Klein, Ludwig Bieberbach (a Nazi mathematician), transformed this one in a pure German mathematician inside his own classification of mathematical types. Thus Bieberbach attributed his own views concerning German mathematics to Klein himself.

In 1934 Bieberbach gave a racial orientation to his mathematician classifications, writing two articles on the subject: "Persoenlichkeitsstruktur und mathematisches Schaffen" ["The Structure of Personality and Mathematical Creation"], and "Stilarten mathematischen Schaffens" ["Styles of Mathematical Creation"]. He was inspired by the racist work of Erich Jaensch, in particular by what the psychologist from Marburg (a convinced Nazi) had written in 1931 in *Grundlagen der menschlichen Erkenntnis* [Foundations of Human Knowledge]. Jaensch did not simply compare 'Germans' and 'not Germans /Jews', using the two capital letters I and S, with I standing for Integrationstypus and S for Strahltypus, but he also analysed a number of possible nuances within them (types I1, I2, I3, I2/I3 etc.) to enable him to reconcile his scheme with historical reality, returning on various occasions in his writings to his classifications to change them.

He then defined the Germanic 'I-types' (Integrationstypus), which 'let the influence of experience stream into them' (Segal, 2003, pp. 362–363) and the S-types (Strahltypus—radiating type), which 'only value those things in Reality which their intellect infers in it'. In the latter group he included (like Jaensch) the French and the Jews—in particular Jacobi, Poincaré, Minkowski and Lejeune-Dirichlet; in the former group he placed Klein with Weierstrass,30 Gauss, Euler and even Dedekind and Hilbert (who 'do show a certain preference for thinking over intuition, but this is distinct from the S-type, who denies the connection to an outer reality that is not mentally constructed'). (Segal, 2003 p. 365). Ultimately he proclaimed: 'I am of the opinion that the whole dispute over the foundations of mathematics is a dispute of contrary psychological types, therefore in the first place a dispute between races. The rise of intuitionism seems to me only a corroboration of this interpretation.'

Bieberbach was dismissed from his teaching post war in 1945 and the Aryan ‘intuition’ disappeared.

L.E.J. BROUWER



Brouwer's first particularly significant comments regarding intuition can be identified in his 1907 doctoral thesis *Grondslagen der wiskunde* (Foundations of Mathematics): 'to exist in mathematics means: to be constructed by intuition' (Brouwer CW I, p. 96) and 'Mathematics is created by a free action independent of experience: it develops from a single aprioristic basic intuition, which may be called invariance in change as well as unity in multitude' (CW I, p. 97). This is a first understanding of intuition as a means of construction whose action is described (in the case of a theorem asserting a property of some mathematical objects) in the following terms:

"Usually mathematics is expressed [...] by means of a chain of syllogisms. But the conceptions which are evoked by the words used in such an , consist in the following: Where mathematical objects are given by their relations to the basic or complex parts of a mathematical structure [this means that the object in question is built in connection with the components to which it is said to be related], we transform these given relations by a sequences of tautologies [i.e. by fixing one's attention to different substructures of the mathematical system] and thus gradually proceed to the relations of the object to other component of the structure". (CW I, p. 72)

In the case of an 'affirmative' theorem which seems to start from a structure defined via certain relations embedded within another structure whose construction is not immediately clear (i.e. it seems to start from mere hypotheses) it happens that:

"One starts by setting up a structure which fulfills part of the required relations, thereupon one tries to deduce from these relations, by means of tautologies, other relations, in such a way that the new relations, combined with those that have not yet been used, yield a system of conditions, suitable as a starting-point for the construction of the required structure."

In the case of a theorem denying that a property belongs to a mathematical entity, the construction comes to an end:

"I simply perceive that the construction no longer goes, that the required structure cannot be imbedded in the given basic structure."(CW I, p. 73)

It should be noted here that for Brouwer mathematics is made up of constructions, that is, it is alinguistic, and based on attempts at constructions, which may succeed or fail. They are 'creative' attempts, i.e. they do not follow any fixed rule. So mathematics does not follow logic, it does not use

logic. Logic records the regularities present in expressions of mathematical constructions, which are carried out to support memory and communicate one's results, with an awareness that there are no guarantees of success in another person's same mathematical construction, and that the emotions accompanying the mathematical experience are inevitably linked to the subject (and hence not repeatable).

The second meaning of intuition is the one that originates basic mathematical entities, first of all natural numbers. Brouwer describes this meaning of intuition as the basic phenomenon, the 'simple intuition of time, in which repetition is possible in the form: 'thing in time and thing again', as a consequence of which moments of life break up into sequences of things which differ qualitatively.'(CW I, p. 53)

In a note he calls this intuition 'intuition of two-ity'. (CW I, p. 97)

It is a priori in that it is independent of experience, while it is not a necessary condition for experience (CW I, p. 70), because mathematics and experience exist independently of each other; but it is a necessary condition of the 'mathematical receptacle of experience'.

In 1918, Brouwer expands the mathematical content of this **intuition**, highlighting the way it is **foundational to the concept of species** ('Unter einer Spezies erster Ordnung verstehen wir eine Eigenschaft welche nur eine mathematische Entitaet besitzen kann [...] Unter einer Spezies zweiter Ordnung verstehen wir eine Eigenschaft welche nur eine mathematische Entitaet oder Spezies erster Ordnung besitzen kann [...] In analoger Weise definieren wir Spezies n- ter Ordnung' - CW I, p. 151) and that **of 'spread'** (spreiding). This is initially described with reference to the universal tree (thought of as a growing structure), i.e. a tree with all possible branches: at each branching point - called 'node' - one assigns either sterilization or a term or nothing. The possibility of assigning a sterilization, which causes the sterilization of the entire branch, is introduced to model the tree, by cutting out the branch; the possibility of assigning nothing, generating finite successions, was used to homologate their construction to that of infinite successions.

Later (after Griss's criticism of his definition of negationi), from his Cambridge lectures on, Brouwer replaced the sterilization procedure with a direct indication to prosecute only certain nodes, i.e. describing the construction of the tree without passing through the universal tree by saying that it has:

1) for initial nodes (of order 1) either all natural numbers or only those not exceeding a certain given m;

2) for nodes of order n+1 (for each n) or all immediate descendants of the node p of order n or only those whose (n+1)-th constituent joined to the constituents of p does not exceed a certain number m_p.

To achieve a spread, to each node either objects or nothing are attached.

Within the production of the spread, during the construction of the tree, Brouwer contemplated freedom of choice in the continuation (and intended each branch as a succession of free choices). Still, 'freedom' encompasses everything and, therefore, can also allow for its progressive restriction and even restriction of restriction.

Brouwer had numerous second thoughts on the subject, but from 1946 onwards he maintained a definitive opinion, saying: "In some former publications of the author restrictions of freedom of future restrictions of freedom, restrictions of freedom of future restrictions of freedom of future restrictions of freedom, and so on were also admitted. But at present the author is inclined to think this admission superfluous and perhaps leading to unnecessary complications." (Brouwer, 1981, p. 13)

Finally, Brouwer rethought his definition of temporal intuition, contextualising it with the original *Weltanschauung* context that he had not been allowed to make explicit in his 1907 thesis: man can find serenity only in the interiority of his own consciousness. He is compelled by karma to go out, but it is appropriate for him to do so only minimally. In particular, scientific work must avoid being applicative and take place inward form: for mathematics, the perfect starting point is the intuition of time. It was at the 1948 conference ("Consciousness, Philosophy, and Mathematics") that Brouwer set out the steps from the inner self to the sciences in greater depth. On that occasion, Brouwer explained temporal intuition within the description of the path of man's consciousness (an unquestionable starting point for him) towards externality: consciousness oscillates between sensation and tranquillity, followed by another sensation and therefore distinguishes between present and past; then it distinguishes itself from both (becoming 'mind'); it identifies complexes of sensations that repeat themselves (if the order never changes, they are called 'things', among which there are human bodies) driven by 'causal attention', i.e. the desire to know and obtain objects.

Mathematics comes into being, when the two-ity created by a move of time is divested of all quality by the subject, and when the remaining empty form of the common substratum of all two-ities, as basing intuition of mathematics, is left to the unlimited unfolding, creating new mathematical entities in the shape of pre-determinately or more or less freely proceeding infinite sequences of mathematical entities previously acquired, and in the shape of mathematical species. (CW I, p. 482)

Brouwer (far from setting up questions of race) tried to impose this type of foundation on the European, and even world, mathematical scene (leading to periods of suspension from research activity due to serious disputes with colleagues), for general well-being,
despite the fact that, at the conference of 1948, he came to support the impossibility of a plurality of minds (while a plurality of bodies can be observed):
"It is not unreasonable to derive this behaviour [the behaviour of individuals in general] from 'reason'. But unreasonable to derive it from 'mind'. For by the choice of this term the subject in its scientific thinking is induced to place in each individual a mind with free-will dependent on this individual, thus elevating itself to a mind of second order experiencing incognizable alien consciousness as sensations. Quod non est. And which moreover would have the consequence that the mind of second order would causally think about the pluralified mind of first order, then cooperatively study the science of the pluralified mind, and in consequence of this study assign a mind of second order with sensation of alien consciousness to other individuals, thus once more elevating itself, this time to a mind of third order. And so on. Usque ad infinitum. […]

In default of a plurality of mind, there is no exchange of thought either. Thoughts are inseparably bound up with the subject. […] By so-called exchange of thought with another being the subject only touches the outer wall of an automaton. This can hardly be called mutual understanding. […] Only through the sensation of the other's soul sometimes a deeper approach is experienced. (CW I, p. 485)

The problem remains open as to how it is possible to be certain that in every human being there is a conscience in which she/he can rest peacefully, but Brouwer mentions neither the problem nor a possible answer.

Arend Heyting

In the thirties, Brouwer's student Arend Heyting came onto the mathematical scene. His first approach was one he would never abandon: building bridges for the sake of understanding and cooperation among mathematicians. He wrote a series of articles on the formal presentation of arithmetic and intuitionistic logic, despite sharing with Brouwer the idea that mathematics is a mental construction and that in it language serves solely expressive, not demonstrative, purposes. Moreover, he took part at the mathematicians meeting in Koenigsberg, the round table on foundational currents, where John von Neumann represented formalism and Rudolf Carnap represented logicism. The atmosphere was one of co-operation (to the extent that individual speakers expressly sought meeting points with the thought of the others) and Heyting entered the stage effortlessly, as a representative of intuitionism, avoiding propaganda.

In his volume *Beweistheorie. Intuitionismus* of 1934 Heyting stated that mathematics has as its only source 'an intuition which sets before our eyes its concepts and conclusions as immediately clear.' (Heyting, 1934, p. 14) It is no more than the faculty of considering concepts and conclusions that habitually occur in our thinking separately. It is a faculty that one must train oneself to exercise, 'a peculiar mental aptitude' that allows mathematics to develop in full autonomy from any philosophical presupposition.

Heyting described intuition in his 1956 book *Intuitionism: an Introduction* as follows: ‘A mathematical construction ought to be so immediate to the mind and its result so clear that it needs no foundation whatsoever. One may very well know whether a reasoning is sound without using any logic; a clear scientific conscience suffices.’ (Heyting, 1956, p. 6).

Then he specified, in the course of various writings, the entities that intuition can attest to. Regarding two-oneness, he wrote: 'We know how to build up the sequence of natural numbers in such a way that we begin to think in terms of a unity, in the same spiritually constructive way that had to be done in forming the observation "a pencil". Then we think "another unit", and finally we think that this last step is repeated again and again. The three concepts "one", "another one" and "again and again one" are sufficient to explain the theory of natural numbers.' (Heyting, CP, pp. 278-279)

Heyting constructed the same mathematical entities as his teacher but he felt the need to specify, with respect to alternative label "choice sequences" for them, that he preferred 'infinitely proceeding sequences' because 'to arrive at the notion of infinitely proceeding sequences, we need not introduce new ideas, in particular the notion of choice' (Heyting, 1956, p. 33), which seemed to him overly linked to the psychology of the subject. In particular, when Brouwer died, he stated that he had glimpsed in the latter's 1948 writing a solipsistic turning point (with a pronounced role accorded individual psychology) that he could not share. That is, in 1948 Brouwer had introduced the expression 'creative subject'.

G.F.C. Griss

Griss had arrived to intuitionism from his own *Weltanschauung* that he had outlined in his 1946 book *Idealistische filosofie*. There, he had based his *Weltanschauung* on the original datum that consciousness grasps by attaining its own fullness: the subject distinguishes himself from the object, but the one has no meaning without the other. Mathematics is the specific way to analyse the original datum that focuses on the subject-object link. For this reason, mathematical objects cannot be thought of independently of a mathematician that produces them: a platonic existence for them is excluded. Griss' *Weltanschauung* had led him to intuitionism. Still, this did not imply a total acceptance of Brouwer's system.

In particular, he criticized Brouwer's definition of negation as a reasoning that ends in a contradiction, i.e. that cannot be carried out, by explaining that: "To assume that a proof is given, while this proof appears to be impossible, is incompatible with the constructive and evidential starting point, because the existence of a proof is identical to the fact that it has been given". (Griss, 1948, 71) Griss criticised the Brouwerian definition of negation, because an intuitionist demonstration must start with something evident and end with something evident. Brouwerian negation had been described as arriving at the proof of the *impossibility* of a construction.

The point at which the proof stops can be considered as evident, because one sees that, metaphorically, one hits a wall, but no status of evidence can be attached to the starting point of the proof, otherwise there would be an evidence that is then disproved: which would remove all foundation for intuitionist mathematics. Hence, Brouwer's definition of negation cannot be considered acceptable within mathematics: it can only be kept at a pre-mathematical stage. A new definition of negation within mathematics is needed. Griss suggested a comparison between two already constructed entities and the realization that one has more properties than the other.

Brouwer responded by constructing a real number which was certainly not zero (i.e. regarding which a negative property was known) but which could not be said to be greater than or less than zero (positive properties), to show that it is not always possible to find a positive substitute for a property defined through disputed negation: he responded to Griss's criticism by arguing that it would be a loss for intuitionist mathematics if the properties defined through the disputed negation were to be eliminated, because some properties would be irretrievably lost.

Heyting did not follow or comment on Brouwer's continuous second thoughts regarding limiting freedom of choice, while he took seriously the doubts that, even from the intuitionist side, one could ever arrive at some of the constructions presented by Brouwer. In particular, reflecting on Griss's critique, he realized that its core was directed against hypothetical constructions which fail, and he grouped together the various types of 'bedingte Konstruktionen' (conditional constructions) present in intuitionist concepts. He began drawing up a list of these in 1949 (CP, pp. 459–460), refining it in 1958 (CP, pp. 560–564, pp. 103–104), and providing a detailed and final version of it in 1962 (CP, p. 641) within a scale of degrees of evidence:

the highest grade is that of such assertions as 2 + 2 =4. 1002 + 2 = 1004 belongs to a lower grade; we show this not by actual counting, but by a reasoning which shows that in general (n+2) + 2 = n + 4. Such general statements about natural numbers belong to a next grade. They have already the character of an implication [...] This level is formalized in the free variable calculus. I shall not try to arrange the other levels in a linear order; it will suffice to mention some notions which by their introduction lower the grade of evidence: 1) The notion of the order type ω, as it occurs in the definition of constructible ordinals. 2) The notion of negation, which involves a hypothetical construction which is shown afterwards to be impossible. 3) The theory of quantification. The interpretation of the quantifiers themselves is not problematical, but the use of quantified expressions in logical formulas is. 4) The introduction of infinitely proceeding sequences. 5) The notion of a species

Heyting stressed that individual intuitionists' willingness to accept hypothetical constructions varies. The starting point is strictly finite mathematics and then one decides how far the arc of mathematical entities acceptable as evident can be stretched. Still, accepting the existence of entities of which we only know the impossibility of non-existence would be very different: that would not be stretching the arc, but going in a completely different direction from the others on the scale. It would be a leap into metaphysical darkness.

Heyting showed a peaceful and benevolent attitude within intuitionism, accepting the various shades of constructability, but this did not mean that he considered platonists as enemies: only, he could not accept the entities they believe in.

He considered this topic also in his unpublished manuscripts, where he mentioned an ascending scale of abstraction, ranging from one's consciousness to real numbers and beyond to God. He specified that some people stop early on and do not accept even very large natural numbers (truly unconstructable by the human mind), and some others believe that there are also Platonic ideas of number or notion-limits for human reason, such as God.

As for he himself, Heyting did not feel up to taking the last step to the top but he understood that others might, on conscience grounds, and he let them do so, without feeling the need to convince them forcibly and, at the same time, declaring that he could not be convinced.

One might approximate this approach to Carnap's principle of tolerance: '[...] das Toleranzprinzip: wir wollen nicht Verbote aufstellen, sondern Festsetzungen treffen. [...] In der Logik gibt es keine Moral. Jeder mag seine Logik, d.h. seine Sprachform aufbauen wie er will' (the principle of tolerance: we don't want to impose prohibitions, but to make determinations. [...] There is no morality in logic. Everyone may construct his logic, i.e. his form of language, as he wishes).(Carnap, 1934, pp. 44–45).

Carnap referred to logic, but we know that logic, according to intuitionism, is the expression of mathematics: therefore, tolerance in logic mirrors tolerance in mathematics. However, it should be noted, from a historical point of view, that Heyting mentioned the principle of tolerance in the fictitious debate at the beginning of this 1956 volume, putting it into the mouth of the representative of the 'formalists' (Heyting, 1956, 2), but did not cite it as the source of his own pluralism.

In addition, we must remember a warning that comes to us from the philosopher Elio Franzini in the context of the Enlightenment legacy: "“The Enlightenment taught, with all its limitations, **tolerance** (a necessary value, and certainly **not sufficient**, which is nevertheless *the basis for its dialogical evolution*)”

If we recall this warning and apply it to our topic, we can state that “tolerance” simply means marking out one’s own territory and those of others in order that each can cultivate their own garden separately and in isolation: it is the premise for, but not yet the definitive step towards, dialogue. Therefore, the word ‘tolerance’ does not fully express Heyting’s attitude: he did not draw furrows in the mathematical ground in order to barricade himself inside his own territory and carry out his work in blissful isolation but encouraged methodological self-awareness during mathematicians’ research and suggested that each identify the most suitable ground for the growth of their seeds, showing an ever lively desire to make their own seeds known to those near and far.

Conclusions

Poincaré cited two types of approach to mathematics: intuitive and analytical, but did not relate these to nationality or ethnicity. He saw them as on a par with hair color: his opinion might be summed up by stating that your mathematical approach is in your DNA. These are flip sides of the same coin, however: the overall view of proof. There are people who see this abstractly and those who see it graphically, but it is, in any case, an overall view. Poincaré considered this overview unteachable, and thus condemned those without it to understanding mathematics but not creating it – if they have sufficient memory – or even to an inability to understand it at all in addition to not creating it if they do not have a powerful memory.

Bieberbach (by referring to a racial mention in F. Klein VI Evanston lecture) distinguished between Germanic 'I-types' (Integrationstypus), who 'let the influence of experience stream into them' (Segal, 2003, pp. 362–363) and 'S-types' (radiating typus), who 'only value those things in reality which their intellect infers in it'. His teaching of mathematics was oriented towards the concreteness of its applications, in order to educate German youth with the type of mathematics suitable to the Aryan race. It was, therefore, a racial not DNA distinction (unlike Poincaré's non-racial DNA concept) which did not, in any case, square well with historical reality, to the point of requiring continuous revisions and the addition of internal nuances in order to reconcile his descriptions of mathematicians with those who really existed.

We have seen that Brouwer did not link intuitionism to race, but made it an obligation for all men in order to practice mathematics without excessively compromising their mystical inner serenity: the purpose of his foundational vision might be said to have been for the sake of good, taking for granted that all men have the same inner lives (while, at the same time, failing to demonstrate the possibility of other minds). Gradually, over the course of his inner considerations on the intuition of time, he discovered the faculty to construct natural numbers, species and free choice sequences, from which he then proceeded to construct all mathematics in a creative way. He followed no specific rules, but rather checked the evidence, the intuitiveness of each step, by experiencing the sense of correctness that also an accountant has when the results "come to him".

Heyting addressed this issue along with other criticisms of the intuitiveness of Brouwerian concepts (e.g., the notion of free-choice sequence), organized the various notions along a scale of degrees of evidence, and admitted (without specifically labelling it) a kind of pluralism within intuitionism, underlining, however, that the distinction between those who call themselves intuitionists (while disagreeing with each other on the admissible mathematical entities) and those who do not

remains clearly visible, because intuitionists limit their acceptance of the existence of the mathematical entities proposed by 'classical' mathematicians.

Unlike Brouwer, moreover, Heyting did not claim that intuitionism was the mathematics to convince others of, but stated that it was possible for some people to believe in the existence of entities (and not only mathematical ones) that were unacceptable for others.

This is the crux of Heyting's logical pluralism (derived from his "mathematical pluralism") and his adherence to a kind of principle of tolerance. 'Tolerance', however, might not be the most appropriate expression, because it does not necessarily involve dialogue between the parties, whereas Heyting desired and sought dialogue. In particular, he wanted to be able to make the other side understand what 'his' mathematics consisted of as well. Hence, the most appropriate expression is 'dialogue'. Heyting proposed logical pluralism and tirelessly sought dialogue. Following his example can be a good educational way to let people become used to look for a dialog in all circumstances of life.

References

Brouwer L.E.J. (1975) *Collected Works vol. 1* (A. Heyting ed.), NorthHolland, Amsterdam.

Brouwer, L. E. J. (1981). *Brouwer's Cambridge Lessons on Intuitionism*. Cambridge U.P.

Carnap, R. (1934). *Die logische Syntax der Sprache*. Springer.

Franchella, M. (1994a) Brouwer and Griss on Intuitionistic Negation. *Modern Logic, 4*(3), 256–265.

Franchella, M. (1994b). Heyting's Contribution to the Change in research into the Foundations of Mathematics. *History and Philosophy of Logic, 15*(1994), 149–172.

Franchella M. (2022) Intuitionism in Mathematics: from Racism to Pluralism, *Philosophia (online 06/04/2022)*

Franzini E. (2009) Elogio dell'Illuminismo, Bruno Mondadori: Milano.
Griss G. F. C. (1946) *Idealistische filosofie*. Van Loghum Slaterus, Arnhem.
Griss G.F.C. (1948) Sur la negation (dans les mathematiques et la logique), *Synthese* VII: 71–74.
Heyting A. (1934) *Mathematische Grundlagenforschung. Intuitionismus. Beweistheorie*, Springer, Berlin.
Heyting, A. (1956). *Intuitionism: An Introduction.* North-Holland.
Heyting, A. (1958a). Blick von der intuitionistischen Warte. *Dialectica, 12*, 332–345.
Heyting A. (1958b) lntuitionism in Mathematics, in *Philosophy in the Mid-century. A Survey*. (R. Klibansky
ed.) La Nuova Italia, Firenze: 101–115.
Heyting, A. (1962). After thirty years, in *Logic* (E. Nagel, P. Suppes, & A. Tarski Eds.), Stanford University Press: Stanford, 194–197
Heyting A. (1980) *Collected papers* (A. Troelstra, J. Niekus, H. van Riemsdijk eds.), Mathematisch Instituut, Amsterdam.

Jaensch, E. R. (1931). *Ȕber die Grundlagen der menschlichen Erkenntnis*. Johann Ambrosius Barth: Leipzig.
Jaensch, E. R. (1938). *Der Gegentypus*. Johann Ambrosius Barth: Leipzig.
Jaensch, E. R., & Althoff, F. (1939). *Mathematisches Denken und Seelenformen*. Johann Ambrosius Barth: Leipzig.
Poincaré, H. (1900). *La valeur de la science*. Flammarion: Paris.
Poincaré, H. (1908). *Science et méthode*. Flammarion: Paris.
Poincare, H. (2014). *The Foundations of Science: Science and Hypothesis, The Value of Science, Science and Method*. Cambridge U.P: Cambridge.
Rowe, D. E. (1985). Felix Klein's "Erlanger Antrittsrede". A transcription with English translation and commentary. *Historia Mathematica, 12*, 123–141. https:// doi. org/ 10. 1016/ 0315- 0860(85) 90003-5
Rowe, D. (1986). 'Jewish mathematics' at Gottingen in the era of Felix Klein. *Isis, 77*(3), 422–449.
Rowe D. (1994) The philosophical views of Klein and Hilbert. In *The intersection of history and mathematics,* (Ch., Sasaki, Ch., Sugiura, M., Dauben, D.W. eds.), Birkhauser, Basel, 187–202.
Vahlen T. (1923) *Wert und Wesen der Mathematik*. Greifswalder Universitatsreden Nr. 9, Greifswald.