Modal Frame Incompleteness.

An Account
through Second Order Logic

Mircea Dumitru
University of Bucharest, Faculty of Philosophy

1 Some basic notions of modal metalogic; frames and validity

- We define three semantic concepts, that of (i) a formula's being *true at a world in a model M*, (ii) of a formula's being *valid in a model M*, and (iii) of a formula's being *valid in a frame F*. To define being true at a world in a model M, we recursively define a relation |= (read: "verifies") as the least relation satisfying the usual conditions.

- We define further two notions of validity: *validity in a model M; validity in a frame F.*

- These definitions concern only single formula validity. However, we can define complementary notions of semantic consequence in the usual way.

- We define now the main metalogical concepts of interest for this paper.
- A deductive system of modal logic is either the system K or a proper extension of K obtained by adding a decidable collection of axiom-sequents to K, at least one of which is not itself K-derivable.
- A deductive system S is *sound with respect to a class of frames K* iff every S-provable sequent is K-valid.
- A deductive system S is *complete with respect to a class of frames* K iff every K-valid sequent is S-provable.
- A deductive system S is *characterized by a class of frames* K iff S is both sound and complete with respect to K, i.e. the S-provable sequents and the K-valid sequents are the same.
- Lastly, a deductive system S is *complete simpliciter* iff there is some class K of frames such that S is characterized by K.

- Since its inception, at the end of 1950s and the beginning of 1960s, the possible worlds semantics has become an enormously successful program. Due to this powerful and flexible formal tool many modal systems got a real and insightful semantic interpretation.

- The methodological success of characterizing modal systems motivated the reasonable hypothesis that every modal deductive system is complete in the absolute sense defined above, i.e. it is characterizable by a class of Kripke-frames.

- Today we know that this hypothesis is false, and we owe this piece of knowledge to the research of some modal logicians, viz. Kit Fine, S. Thomason, Johan Van Benthem, G. Boolos and G. Sambin, M. Cresswell.

- My aim here is just to show, first, what a semantic incomplete system looks like, and then to look for an explanation of this interesting and also curious semantic phenomenon.

- One way of putting this fact of there existing incomplete propositional modal logics is to say that there is a class of frames F that characterize a logic L that is not axiomatizable.

- A similar phenomenon occurs in second order classical logic, where one quantifies over subsets of the domain as well as over individuals. Classical second order validity is not axiomatizable; it too displays incompleteness aspects.

- To that problem Henkin offered a solution, which he called *general models*. In these, set quantifiers are restricted to a designated collection of subsets of the domain, and do not range over all subsets.

- Validity with respect to general models is axiomatizable. Henkin's general models were, in fact, the inspiration for the introduction of general frames into modal semantics.
- Against this background, the gist of this paper is to give a second order based explanation of modal incompleteness.
- The technical apparatus which is deployed in my argumentation builds upon a case of the embedding of modal logic in second order logic. The leading concept is that modal incompleteness is to be explained in terms of the incompleteness of standard second order logic, since modal language is basically a second order language.
- That is, the paper shows that modal frame incompleteness is a kind of exemplification of classic second order incompleteness.

2 The Incomplete System VB

- We shall show, following van Benthem, that a certain system of modal logic, VB (to honor Van Benthem) is incomplete, i.e. it is a system which is characterized by no class of frames K. So a better tag for such a system would be “uncharacterizable system”.
- The real form of uncharacterizability results is that of a conditional: “if system S is sound with respect to K then S is not complete with respect to K”.
- We first define the system K* to be the system K plus an additional axiom-sequent, viz. $L(A \rightarrow B) \rightarrow (LA \rightarrow LB)$ + the axiom sequent $MLA \vee LA$.

- I state the following result which is a Sahlqvist case:
- Claim 2.1: K* is characterized by the class of frames in which every world is either a dead end or else is one step removed from a dead end; w is a dead end if it can see no world.
- We define now the system VB to be the system K plus the axiom-sequent MLA v L(L(LB → B) → B).

- The proof that VB is incomplete proceeds in two steps.
- First we show Step 1: Every frame for VB is a frame for K*.
- Then we show Step 2: MLA ∨ LA is not a theorem of VB.
- We have to make clear why is this establishing the incompleteness of VB. The reason is as follows:
- Suppose K is a class of frames with respect to which VB is sound. Then, once we have established Step 1, we may conclude that there are no counterexamples to K*-sequents based on frames in K. In that case, |=K MLA ∨ LA.
- Hence, there is a sequent which is valid in K which, granted Step 2, is not derivable in VB.
- So VB is incomplete with respect to K (the class of frames with respect to which it is sound).

3 Semantics for the language of second-order logic

- Basically, the concept of interpretation in second-order logic is similar to the one in first-order logic. For my argument I need the two distinct kinds of interpretation for the languages of second-order logic: the *standard interpretation* for the languages of second-order logic, and then the *Henkin (or general) interpretation* for second order logic.

- Remark 3.1: The whole difference between standard semantics and Henkin semantics can be accounted for in terms of the different meanings that are attached to the phrase ‘every assignment’ when defining, Tarski-style, the notion of quantification with regards to functions or to predicates.

- In the case of standard semantics an assignment to an n-place predicate variable and to an n-place function variable makes the variables range over the *whole powerset* of Dn, and over the collection of *all functions* from Dn to D, respectively.

- Whereas in the case of the Henkin semantics the collection of assignments *may be restricted* to those assignments only that assign members of different D*n, where D*n ⊆ Dn, and Fn, where Fn ⊆ Dn × D, to the higher-order variables.

4 Explaining incompleteness in modal logic

- Our semantics for modal logic is essentially a semantics for second-order monadic predicate logic (with a single binary relation constant R).
- If we inspect our definition of validity in a frame, we see that for a given wff to be valid in F it must be true in every world in every model based on F.
- The phrase "*every model* based on F" is a universal quantifier over assignments of subsets of W to the sentence-letters of the modal language.
- And since in the canonical translation of LSML into that language of second-order monadic predicate logic a sentence letter of the former becomes a monadic predicate of the latter, the force of "*every model based* on F" is intuitively *no matter what subsets of W are assigned to the corresponding monadic predicates*.
- Hence, the quantification over models in the modal semantics can be captured by a second-order universal quantifier.

- For example, the statement
- $\models_F \quad L(P \& Q)$
- says that for every $w \in W_F$, every world w can see satisfies P and satisfies Q, no matter what properties (subsets of W_F) are assigned to P and to Q.
- So, in second-order monadic logic,
- $\models_F L(P \& Q)$ can be written
- $F' \models \forall P' \forall Q' \forall w \forall u(Rwu \rightarrow (P'u \& Q'u))$
- in which we have changed "$\models_F$ " into "$F \models$" to indicate that the pair (W,R) is being regarded as an interpretation for a second-order language with a single binary relation constant R.

- To carry out the details of this reductive argument we have to show how the language and the semantics of sentential modal logic can be mapped into the language and the semantics of second order logic.
- To this purpose we need a collection of recursive rules of translation (schemata) that will take formulae (wffs) of the language of sentential modal logic as input and will yield the corresponding formulae of the language of second-order logic as output.
- What we look for here is a language in which the translation that is carried over is instrumental for the explanation that is sought here, viz. incompleteness in modal logic as a second-order phenomenon.
- That also backs the concept that modal frame incompleteness is a kind of exemplification of second order incompleteness. And it turns out that what we need is a second order language that for obvious reasons will be called the language of canonical translation (LCT).
- In a few words, what we are after here is the bringing about of a mechanism that will allow us to recast the whole apparatus needed to prove the incompleteness of the system VB into the terms that are proper to second order logic.

- However, it is not only formulae of LSML that have to be mapped into corresponding formulae of LSOL. For to carry out the attempted explanation of modal incompleteness, we also need a way of reconfiguring the modal possible world semantics and the main metalogical modal notions definable within that frame as a second order semantics, and second order metalogical notions, respectively.

- It is worth keeping in mind that with respect to modal languages two different modal semantic systems can be constructed, viz. one which is based on the notion of Kripke-frame, and a second one which is based on the notion of General-frame.

- The main modal concept of interest for the issue of completeness vs. incompleteness, viz. the notion of *a modal formula's being valid in a frame*, gets the well-known definition "true in every world in every model based on a given frame".

- And of course, the definition will differ according to whether the frame in question is a Kripke-frame or a General-frame.

- However, this “recasting” does not really “explain” why every frame for VB is a frame for K*, since it merely restates our earlier proof in a different language.
- But it allows us to relate the incompleteness of VB to the non-existence of a sound and complete set of inference rules for second-order logic.
- Remark 4.5: The point is not that in second-order logic VB $\nvdash_2$ K*. For deductively, there is no one thing which is second order logic.
- Instead, there are various significantly different sound deductive systems. And though none of them is complete, there is certainly some collection of rules determining a deductive consequence relation $\vdash_2$ such that VB $\vdash_2$ K*.
- For instance, trivially, we could introduce a second order system of deduction in which the step from any instance of VB to a corresponding instance of K* is a primitive rule.

- So what, then, is the explanation of the incompleteness of VB?
- To make some progress into this issue it is useful to think of modal systems other than K as theories of modality, and K as the logic. Thus, where before we would have written $A \vdash_T MA$, treating the T-sequent as part of the logic, i.e. as part of the definition of a new deductive consequence relation $\vdash_T$, we will now write instead that
- $A, L\neg A \rightarrow \neg A \vdash_K MA$.

- This gives rise to a notion of *formula completeness* that is the counterpart of the notion of system completeness:

- A formula $\boldsymbol{\sigma}$ of LSML is said to be *complete* iff all the semantic consequences of it and its substitution instances (relative to auxiliary premises and the class of all frames) are derivable from it in K. In symbols: $\boldsymbol{\sigma}$ is complete iff whenever $\boldsymbol{\Gamma}$, $\boldsymbol{\sigma}^*$ $\models \gamma$ then $\boldsymbol{\Gamma}$, $\boldsymbol{\sigma}^*$ $\models_K \gamma$, where $\boldsymbol{\sigma}^*$ is a substitution instance of $\boldsymbol{\sigma}$.

- The result that VB is an incomplete system becomes in this terminology the result that MLA ∨ L(L(LB → B) → B) is an incomplete formula, because
- MLA ∨ L(L(LB → B) → B) $\models$ MLA ∨ LA
- but
- L(L(LB → B) → B) $\nvdash$K MLA ∨ LA.

- The explanation of the incompleteness of VB is then (not that there is no second order logic in which VB entails K*, but rather) that the deductive consequence relation $\vdash_K$ of modal systems (of theories of modality) is significantly weaker than $\vdash_{p2}$ in a precise sense:

- Definition 4.6: Let L and L* be two languages and let T (from 'translation') be a function T: L ⟜ L*, from the sentences of L into the sentences of L*. Then $\vdash^*$ is an L*-consequence relation which is said to be a *conservative extension* of an L-consequence relation $\vdash$, with respect to T, provided the following holds for any set of L-sentences Σ and any L-sentence $\boldsymbol{\sigma}$:

- T(Σ) $\vdash^*$ T($\boldsymbol{\sigma}$) only if $\Sigma \vdash \boldsymbol{\sigma}$.

- It follows that a conservative consequence relation is one in the new language which does not allow sequents to be proven unless they are the translation of provable sequents from the original language.

- Since we already know that

- $MLA \lor L(L(LB \rightarrow B) \rightarrow B) \nvdash_K MLA \lor LA$,

- and we have just seen that $VB \vdash_{p2} K^*$, this shows that $\vdash_{p2}$ is non-conservative over $\vdash_K$.

- Its extra strength comes in part from the fact that the second order variables can substitute for, and be substituted by, any first order formula with one free variable.
- But it is perfectly conceivable that there should be interpretations with a first order definable set of worlds that is not the worldset of any modal sentence; for example, in any transitive model where there are two worlds u and v which see and are seen by the same worlds, and which make the same atomic sentences true, no modal sentence can have W – {u} or W – {v} as its worldset. But "_ ≠ u" is still a perfectly acceptable first order formula which is satisfied by all and only the members of W – {u}.
- In general, then, in predicative monadic second-order logic we can reason with statements that cannot even be expressed in LSML, which allows us to prove sequents in the former logic which are not provable modally.

- The overall moral, then, is that *the system VB is uncharacterizable because of the lack of expressive power of LSML as compared to the expressive power of LSOL.*

- Thus, as the case that I presented in this paper shows, we can reason in predicative second order logic with formulae that LSML has no power to express.

- That is the main rationale for there being the case that the sequent Fsot[VB] ⊢p2 Fsot[K*] is provable in second order logic, whereas its modal version, viz. VB ⊢K K* is not provable modally.

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