

The spectrum problem for Abelian ℓ -groups and MV-algebras

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Io vo gridando pace, pace, pace

F. Petrarca, Canzoniere, CXXVIII

MV-algebras

- ▶ MV-algebras are algebraic models of the Łukasiewicz logic.
- ▶ (Mundici-Riečan paper, page 871, line 6) They can be considered as a non-commutative generalization of Boolean algebras, using the language of C^* -algebras.
- ▶ They can be seen also as non-idempotent generalizations of Boolean algebras.

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MV-algebras

They are structures $(A, \oplus, \neg, 0)$ such that

1) $(A, \oplus, 0)$ is a commutative monoid;

2) $\neg\neg x = x$;

3) $x \oplus 1 = 1$, where $1 = \neg 0$;

4) $x \rightarrow (x \rightarrow y) = y \rightarrow (y \rightarrow x)$, where $x \rightarrow y = (\neg x) \oplus y$
(Mangani axiom).

Mundici equivalence

There is a categorial equivalence between MV-algebras and Abelian ℓ -groups with a strong unit.

Di Nola-Lettieri equivalence

There is a categorial equivalence between Abelian ℓ -groups and perfect MV-algebras (those consisting only of infinitesimals and their complements).

The order of an MV-algebra

$x \leq y$ if there is z such that $y = x \oplus z$.
This order is a distributive lattice.

Ideals

An ideal I of A is a subset closed under sum and closed downwards.

An ideal I is prime if $x \wedge y \in I$ implies $x \in I$ or $y \in I$.

Ideals correspond to congruences.

The Zariski topology of the prime spectrum

$\text{Spec}(A)$ is the set of all prime ideals of A .

The basic opens are $O_a = \{P \in \text{Spec}(A) \mid a \notin P\}$, where $a \in A$.

The spectrum problem

Which topological spaces are homeomorphic to $\text{Spec}(A)$ for some MV-algebra A ?

(Daniele Mundici, Advanced Lukasiewicz calculus and MV-algebras, page 235, problem 2)

The Belluce lattice

Let A be an MV-algebra.

There is an isomorphism between

- ▶ the lattice of principal ideals of A , and
- ▶ the lattice of compact open subsets of $\text{Spec}(A)$.

This lattice is known as the Belluce lattice of A , $\beta(A)$.

The spectra of A and $\beta(A)$ are homeomorphic.

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The results of Wehrung

Countable Belluce lattices are definable in first order logic.
General Belluce lattices are not definable, even in infinitary first order logic.

Solved variants of the spectrum problem

Boolean algebras (Stone)

Commutative rings (Hochster, spectral spaces)

The order of $\text{Spec}(A)$ under inclusion (Cignoli-Torrens)

Maximal spectrum (Marra-Spada)

Preliminaries 1

Recall that a topological space X is called *generalized spectral* if it is sober (i.e., every irreducible closed set is the closure of a unique singleton) and the collection of all compact open subsets of X forms a basis of the topology of X , closed under intersections of any two members. If, in addition, X is compact, we say that it is *spectral*.

Preliminaries 2

A join homomorphism $f : K \rightarrow L$ between join semilattices is called *closed* if whenever $a_0, a_1 \in K$ and $b \in L$, if $f(a_0) \leq f(a_1) \vee b$, then there is $x \in K$ such that $a_0 \leq a_1 \vee x$ and $f(x) \leq b$.

Cylinder polyhedra

Infinite dimensional analogs of polyhedra.

We define *cylinder (rational) polyhedron* in a hypercube $[0, 1]^X$ a subset of $[0, 1]^X$ of the form

$$C_X(P_0) = \{f \in [0, 1]^X \mid f|_Y \in P_0\}$$

where Y is a finite subset of X and $P_0 \subseteq [0, 1]^Y$ is a rational polyhedron. Here $f|_Y$ denotes the function f restricted to Y , that is, $f|_Y = f \circ j$, where $j : Y \rightarrow X$ is the inclusion map.

The main results

- 1) The Belluce lattice of the free MV-algebra over any set X is dually isomorphic to the lattice of cylinder polyhedra in $[0, 1]^X$.
- 2) A lattice is the Belluce lattice of an MV-algebra if and only if it is a closed surjective image of the Belluce lattice of a free MV-algebra.
- 3) A topological space X is the spectrum of an MV-algebra if and only if X is spectral and the lattice $K(X)$ of its compact open sets is the Belluce lattice of an MV-algebra. These lattices are characterized in 2).

The case of abelian ℓ -groups

This case is analogous to the one of MV-algebras up to replacing polyhedra with cones.

Freely spectral spaces (?)

Just a fancy name for prime spectra of free MV-algebras (maximal spectra of free MV-algebras are hypercubes).

Their lattices of compact opens are partially axiomatized in the paper.

Ongoing work

Consider presented MV-algebras $A = F/I$, where F is free and I is an ideal.

In the Zariski topology of $\text{Spec}(F)$, $C(I) = \{P \in \text{Spec}(F) \mid I \subseteq P\}$ is closed.

This makes us conjecture that presented MV-algebras are categorically equivalent to a category of topological spaces (non full) of closed subsets of F where F ranges over free MV-algebras.

Sheaf theoretic applications

Dubuc-Poveda decompose every MV-algebra into a sheaf of MV-chains over its prime spectrum.

Our construction gives us information about the base space (the prime spectrum).

Topos theoretic applications

Prime ideals and MV-chains are tightly related: A/P is a chain if and only if P is prime.

For every MV-algebra A , (A, \wedge, \oplus) is an additively idempotent semiring.

(tropical) chain semirings occur as stalks of the structure sheaf of a semiringed topos called the arithmetic site (see A. Connes and C. Consani, Geometry of the arithmetic site, Adv. Math. 291 (2016), 274–329.)

Thank you!

We stand with Ukraine!