Admissibility of Π_2-Inference Rules: interpolation, model completion, and contact algebras

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Logic4Peace, 22 April 2022

Π_2-rules

Definition

An inference rule ρ is a Π_2-rule if it is of the form

$$\frac{F(\underline{\phi}/\underline{x}, \underline{y}) \to \chi}{G(\underline{\phi}/\underline{x}) \to \chi}$$

where $F(\underline{x}, \underline{y}), G(\underline{x})$ are propositional formulas.

We say that θ is obtained from ψ by an application of the rule ρ if

$$\psi = F(\underline{\phi}/\underline{x}, \underline{y}) \to \chi \text{ and } \theta = G(\underline{\phi}/\underline{x}) \to \chi,$$

where $\underline{\phi}$ is a tuple of formulas, χ is a formula, and $\underline{y}$ is a tuple of propositional letters not occurring in $\underline{\phi}$ and χ.

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Let $\mathcal{S}$ be a propositional modal system. We say that the rule ρ is admissible in $\mathcal{S}$ if $\vdash_{\mathcal{S}+\rho} \phi$ implies $\vdash_{\mathcal{S}} \phi$ for each formula ϕ.

First method

Conservative extensions

We say that $\phi(\underline{x}) \wedge \psi(\underline{x}, \underline{y})$ is a conservative extension of $\phi(\underline{x})$ in $\mathcal{S}$ if

$$\vdash_{\mathcal{S}} \phi(\underline{x}) \wedge \psi(\underline{x}, \underline{y}) \rightarrow \chi(\underline{x}) \quad \text{implies} \quad \vdash_{\mathcal{S}} \phi(\underline{x}) \rightarrow \chi(\underline{x})$$

for every formula $\chi(\underline{x})$.

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for every formula $\chi(\underline{x})$.

Theorem

If $\mathcal{S}$ has the interpolation property, then a Π_2-rule ρ is admissible in $\mathcal{S}$ iff $G(\underline{x}) \wedge F(\underline{x}, \underline{y})$ is a conservative extension of $G(\underline{x})$ in $\mathcal{S}$.

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Therefore, if $\mathcal{S}$ has the interpolation property and conservativity is decidable in $\mathcal{S}$, then Π_2-rules are effectively recognizable in $\mathcal{S}$.

Corollary

The admissibility problem for Π_2-rules is

- NEXPTIME-*complete in* K *and* S5*;*
- *in* EXPSPACE *and* NEXPTIME-*hard in* S4*.*

Second method

Uniform interpolants

An S5-modality $[\forall]$ is called a universal modality if

$$\vdash_{\mathcal{S}} \bigwedge_{i=1}^{n} [\forall](\phi_i \leftrightarrow \psi_i) \rightarrow (\Box[\phi_1, \ldots, \phi_n] \leftrightarrow \Box[\psi_1, \ldots, \psi_n])$$

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If $\phi(\underline{x}, \underline{y})$ is a formula, its right global uniform pre-interpolant $\forall_{\underline{x}}\phi(\underline{y})$ is a formula such that for every $\psi(\underline{y}, \underline{z})$ we have that

$$\psi(\underline{y}, \underline{z}) \vdash_{\mathcal{S}} \phi(\underline{x}, \underline{y}) \ \text{ iff } \ \psi(\underline{y}, \underline{z}) \vdash_{\mathcal{S}} \forall_{\underline{x}}\phi(\underline{y}).$$

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Theorem

Suppose that $\mathcal{S}$ has uniform global pre-interpolants and a universal modality $[\forall]$. Then a Π_2-rule ρ is admissible in $\mathcal{S}$ iff

$$\vdash_{\mathcal{S}} [\forall]\forall_{\underline{y}}(F(\underline{x}, \underline{y}) \rightarrow z) \rightarrow (G(\underline{x}) \rightarrow z).$$

Third method

Simple algebras and model completions

To a Π_2-rule we associate the first-order formula

$$\Pi(\rho) := \forall \underline{x}, z\Big(G(\underline{x}) \not\leq z \Rightarrow \exists \underline{y} : F(\underline{x}, \underline{y}) \not\leq z\Big).$$

Theorem

Suppose that $\mathcal{S}$ has a universal modality. A Π_2-rule ρ is admissible in $\mathcal{S}$ iff for each simple $\mathcal{S}$-algebra $\mathcal{B}$ there is a simple $\mathcal{S}$-algebra $\mathcal{C}$ such that $\mathcal{B}$ is a subalgebra of $\mathcal{C}$ and $\mathcal{C} \models \Pi(\rho)$.

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In the presence of a universal modality, an $\mathcal{S}$-algebra is simple iff

$$[\forall]x = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

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$$[\forall]x = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $\mathcal{S}$-algebras form a discriminator variety. Therefore, the variety of $\mathcal{S}$-algebras is generated by the simple $\mathcal{S}$-algebras.

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Theorem

Suppose that $\mathcal{S}$ has a universal modality and let $T_{\mathcal{S}}$ be the first-order theory of the simple $\mathcal{S}$-algebras. If $T_{\mathcal{S}}$ has a model completion $T^{\star}_{\mathcal{S}}$, then a Π_2-rule ρ is admissible in $\mathcal{S}$ iff $T^{\star}_{\mathcal{S}} \models \Pi(\rho)$ where

$$\Pi(\rho) := \forall \underline{x}, z \Big(G(\underline{x}) \not\leq z \Rightarrow \exists \underline{y} : F(\underline{x}, \underline{y}) \not\leq z \Big).$$

The symmetric strict implication calculus and contact algebras

Definition (G. Bezhanishvili, N. Bezhanishvili, T. Santoli, Y. Venema (2019))

The symmetric strict implication calculus S^2IC is given by the axioms

(A0) $[\forall]\phi \leftrightarrow (\top \rightsquigarrow \phi)$,
(A1) $(\bot \rightsquigarrow \phi) \wedge (\phi \rightsquigarrow \top)$,
(A2) $[(\phi \vee \psi) \rightsquigarrow \chi] \leftrightarrow [(\phi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi)]$,
(A3) $[\phi \rightsquigarrow (\psi \wedge \chi)] \leftrightarrow [(\phi \rightsquigarrow \psi) \wedge (\phi \rightsquigarrow \chi)]$,
(A4) $(\phi \rightsquigarrow \psi) \rightarrow (\phi \rightarrow \psi)$,
(A5) $(\phi \rightsquigarrow \psi) \leftrightarrow (\neg\psi \rightsquigarrow \neg\phi)$,
(A8) $[\forall]\phi \rightarrow [\forall][\forall]\phi$,
(A9) $\neg[\forall]\phi \rightarrow [\forall]\neg[\forall]\phi$,
(A10) $(\phi \rightsquigarrow \psi) \leftrightarrow [\forall](\phi \rightsquigarrow \psi)$,
(A11) $[\forall]\phi \rightarrow (\neg[\forall]\phi \rightsquigarrow \bot)$,

and modus ponens (for $\rightarrow$) and necessitation (for $[\forall]$).

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Let v be a valuation into a topological space X that maps each propositional variable to a regular open of X. We can extend each valuation on all formulas as follows

$$
\begin{aligned}
v(\bot) &= \varnothing \\
v(\top) &= X \\
v(\phi \wedge \psi) &= v(\phi) \cap v(\psi) \\
v(\phi \vee \psi) &= \mathrm{int}(\mathrm{cl}(v(\phi) \cup v(\psi))) \\
v(\neg \phi) &= \mathrm{int}(X \setminus v(\phi)) \\
v(\phi \rightsquigarrow \psi) &= \begin{cases} X & \text{if } \mathrm{cl}(v(\phi)) \subseteq v(\psi), \\ \varnothing & \text{otherwise.} \end{cases}
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Theorem (G. Bezhanishvili, N. Bezhanishvili, T. Santoli, Y. Venema (2019))

$\vdash_{\mathsf{S^2IC}} \phi$ *iff* $v(\phi) = X$ *for every compact Hausdorff space* X *and* v.

The algebras associated with S^2IC are called strict implication algebras.

When a strict implication algebra is simple, $\rightsquigarrow$ becomes a characteristic function of a binary relation. They correspond exactly to contact algebras.

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Definition

A contact algebra is a boolean algebra equipped with a binary relation $\prec$ satisfying the axioms:

(S1) $0 \prec 0$ and $1 \prec 1$;
(S2) $a \prec b, c$ implies $a \prec b \wedge c$;
(S3) $a, b \prec c$ implies $a \vee b \prec c$;
(S4) $a \leq b \prec c \leq d$ implies $a \prec d$;
(S5) $a \prec b$ implies $a \leq b$;
(S6) $a \prec b$ implies $\neg b \prec \neg a$.

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An axiomatization is given by the following three sentences.

$$\forall a, b_1, b_2\ (a \neq 0\ \&\ (b_1 \vee b_2) \wedge a = 0\ \&\ a \prec a \vee b_1 \vee b_2 \Rightarrow \exists a_1, a_2\ (a_1 \vee a_2 = a\ \&\ a_1 \wedge a_2 = 0\ \&\ a_1 \neq 0\ \&\ a_2 \neq 0\ \&\ a_1 \prec a_1 \vee b_1\ \&\ a_2 \prec a_2 \vee b_2))$$

$$\forall a, b\ (a \wedge b = 0\ \&\ a \not\prec \neg b \Rightarrow \exists a_1, a_2\ (a_1 \vee a_2 = a\ \&\ a_1 \wedge a_2 = 0\ \&\ a_1 \not\prec \neg b\ \&\ a_2 \not\prec \neg b\ \&\ a_1 \prec \neg a_2))$$

$$\forall a\ (a \neq 0 \Rightarrow \exists a_1, a_2\ (a_1 \vee a_2 = a\ \&\ a_1 \wedge a_2 = 0\ \&\ a_1 \prec a\ \&\ a_1 \not\prec a_1))$$

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The following Π_2-rule

$$\frac{(p \rightsquigarrow p) \wedge (\phi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi) \to \chi}{(\phi \rightsquigarrow \psi) \to \chi}$$

corresponds to the zero-dimensionality of the space.

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Using the axiomatization of $\mathrm{Con}^\star$ it is easy to show that it is admissible in $\mathrm{S^2IC}$.

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Using the axiomatization of $\mathsf{Con}^{\star}$ it is easy to show that it is admissible in $\mathsf{S^2IC}$.

Therefore, $\mathsf{S^2IC}$ is complete wrt Stone spaces.

Theorem (G. Bezhanishvili, N. Bezhanishvili, T. Santoli, Y. Venema (2019))

$$\vdash_{\mathsf{S^2IC}} \phi \ \textit{ iff } \ v(\phi) = X \ \textit{ for every Stone space } X \textit{ and } v.$$

THANK YOU!

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