$\lambda\rho$-products

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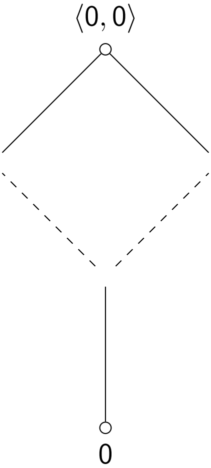
Jipsen-Montagna example

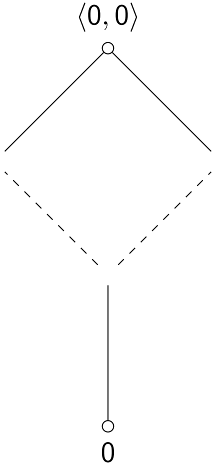
- Start with $(\mathbb{Z}; \leq, +, 0)$ as an ℓ-group.
- Take $\mathbb{Z} \times \mathbb{Z}$ and another copy of $\mathbb{Z}$; extend the natural order on $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z}$ by putting $\mathbb{Z} \times \mathbb{Z}$ on top of $\mathbb{Z}$.
- Truncate to the interval $[0, \langle 0, 0 \rangle]$.
- Products in the top part are as in $\mathbb{Z} \times \mathbb{Z}$.
- All other products are defined by

$$\langle x, y \rangle \cdot i = \max\{x + i, 0\}$$
$$i \cdot \langle x, y \rangle = \max\{y + i, 0\}$$
$$i \cdot j = 0$$



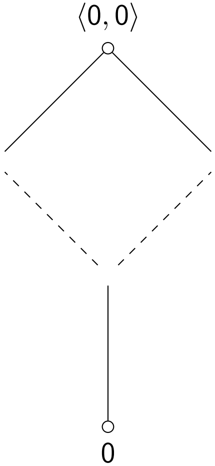
Example generalised

- The algebra you get is in fact a pseudo BL-algebra, a model of a noncommutative version of fuzzy logic.



Example generalised

- The algebra you get is in fact a pseudo BL-algebra, a model of a noncommutative version of fuzzy logic.
- Can also be described as follows:
  - Take a two element semigroup $\{a, b\}$ satisfying $a^2 = a$ and $uv = b$ for all other products.
  - Graft $\mathbb{Z} \times \mathbb{Z}$ into a by and $\mathbb{Z}$ into b.
  - Fix a set of maps λ and ρ between the sets of coordinates, say $I[a] = \{0, 1\}$ and $I[b] = \{0\}$, telling us which coordinate to take for which product.



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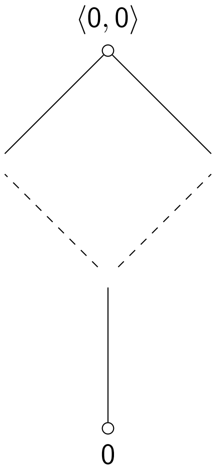
    - Fix a set of maps λ and ρ between the sets of coordinates, say $I[a] = \{0, 1\}$ and $I[b] = \{0\}$, telling us which coordinate to take for which product.
- *Generalises to an arbitrary power of an ℓ-group upstairs, and another one downstairs.*



Example formalised

Then $\langle x, y\rangle \cdot i$ can be presented as

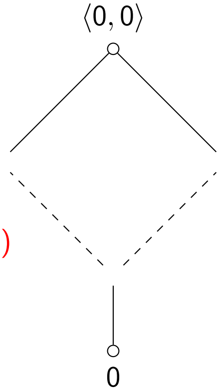
$$((\langle x, y\rangle, a) \cdot (i, b) = ((\langle x, y\rangle \circ \lambda[a, b]) \cdot (i \circ \rho[a, b]), ab)$$

where

- $\lambda[a, b] \colon I[ab] \to I[a]$ is given by $\lambda[a, b](0) = 0$,
- $\rho[a, b] \colon I[ab] \to I[b]$ is given by $\rho[a, b](0) = 0$,
- Calculating the product yields

  $$((\langle x, y\rangle \circ \lambda[a, b]) \cdot (i \circ \rho[a, b]), ab) = (x + i, ab)$$

  which is precisely what we want.
- $\lambda[b, a] \colon I[ba] \to I[b]$ is the identity, of course.

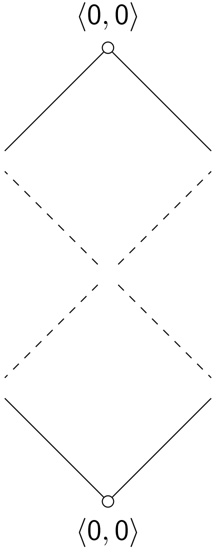


Another example: wreath product

- Now take $\mathbb{Z} \times \mathbb{Z}$ on top of a copy of $\mathbb{Z} \times \mathbb{Z}$; extend the natural order on $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z}$ by putting $\mathbb{Z} \times \mathbb{Z}$ on top of $\mathbb{Z}$.
- Truncate appropriately.
- Set the products between the top and the bottom parts to be

$$\langle a, b \rangle \cdot \langle i, j \rangle = \max\{\langle a + j, b + i \rangle, \langle 0, 0 \rangle\}$$
$$\langle i, j \rangle \cdot \langle a, b \rangle = \max\{\langle a + i, b + j \rangle, \langle 0, 0 \rangle\}$$

- The algebra obtained here is isomorphic to a truncation of a subgroup of the antilexicographically ordered wreath product $\mathbb{Z} \wr \mathbb{Z}$.



$\lambda\rho$-product: intuitions

- Forget all structure except multiplication.
- Take some magma $\mathbf{S}$.
- Let $(I[s])_{s\in S}$ be a system of sets indexed by the elements of S (sets of coordinates, one for each $s \in S$).
- Take a system of maps between the sets of coordinates.
  $\boldsymbol{\lambda} = (\lambda[a,b]\colon I[ab] \to I[a])_{(a,b)\in S\times S}$ and
  $\boldsymbol{\rho} = (\rho[a,b]\colon I[ab] \to I[a])_{(a,b)\in S\times S}$.
- Next, take any magma $\mathbf{H}$, and graft $H^{I[s]}$ into each $s \in S$.
- Define product on $\biguplus_{s\in S} \mathbf{H}^{I[s]}$ as we saw in the example:

$$(\langle x,y\rangle, a)\cdot(i,b) = ((\langle x,y\rangle \circ \lambda[a,b])\cdot(i\circ\rho[a,b]), ab)$$

$\lambda\rho$-product: formal definition

Definition

Let **S** be a magma and let

$$\mathcal{S} = \big(\langle \lambda[a,b], \rho[a,b]\rangle \colon I[ab] \to I[a] \times I[b]\big)_{(a,b)\in S^2}$$

be a system of sets and maps indexed by the elements of S^2. Let **H** be a magma. We define a groupoid $\mathbf{H}^{[\mathcal{S}]} = (H^{[\mathcal{S}]}; \star)$, by putting

- $H^{[\mathcal{S}]} = \biguplus_{a\in S} H^{I[a]} = \{(x,a) \colon a \in S,\ x \in H^{I[a]}\}$, and
- $(x,a) \star (y,b) = \big((x \circ \lambda[a,b]) \cdot (y \circ \rho[a,b]), ab\big)$.

where $\cdot$ is the product in **H** and the product in **S** is written as concatenation.

We call $\mathbf{H}^{[\mathcal{S}]}$ a $\lambda\rho$-product.

Semigroups

If **S** and **H** are semigroups, one may want $\mathbf{H}^{[\mathcal{S}]}$ to be a semigroup, too.

Definition

Let **S** be a semigroup, and let

$$\mathcal{S} = \big(\langle \lambda[a,b], \rho[a,b] \rangle \colon I[ab] \to I[a] \times I[b]\big)_{(a,b)\in S^2}$$

be a system of sets and maps satisfying the following conditions

(α) $\lambda[a,b] \circ \lambda[ab,c] = \lambda[a,bc]$

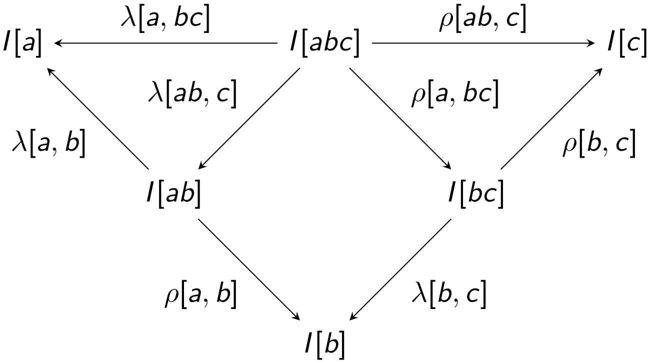
(β) $\rho[b,c] \circ \rho[a,bc] = \rho[ab,c]$

(γ) $\rho[a,b] \circ \lambda[ab,c] = \lambda[b,c] \circ \rho[a,bc]$

Then we call $\mathcal{S}$ a $\lambda\rho$-system.

Diagram for associativity

The conditions α, β and γ amount to the commutativity of this diagram.



$\lambda\rho$-product: associativity

Theorem

Let **S** *be a magma and let*

$$\mathcal{S} = \left(\langle\lambda[a,b], \rho[a,b]\rangle : I[ab] \to I[a] \times I[b]\right)_{(a,b)\in S^2}$$

be a system of sets and maps indexed by the elements of S^2*.*
Then, the following are equivalent.

1. $\mathbf{H}^{[\mathcal{S}]}$ *is a semigroup, for any semigroup* **H**.
2. **S** *is a semigroup and* $\left(\langle\lambda[a,b], \rho[a,b]\rangle : I[ab] \to I[a] \times I[b]\right)_{(a,b)\in S^2}$ *is a* $\lambda\rho$*-system over* **S** *(i.e., satisfies* α, β *and* γ*).*
3. **S** *is a semigroup and there exists a nontrivial semigroup* **H** *such that* $\mathbf{H}^{[\mathcal{S}]}$ *is a semigroup.*

Flip-flop monoid can be decomposed: an example

Let $\mathbf{2} = (\{0,1\}, \vee)$ be the two-element join-semilattice, and let $\mathcal{Z}$ be the $\lambda\rho$-system over $\mathbf{2}$, defined by putting

1. $I[0] = \{0\}$, $I[1] = \{0,1\}$,
2. $\lambda[1,0] = \rho[0,1] = \lambda[1,1] = id_{I[1]}$ and $\rho[1,1] = \overline{0}$.

This defines a unique $\lambda\rho$-system, since the remaining maps all have range $\{0\}$. It is easy to show that the semigroup $\mathbb{Z}_2^{[\mathcal{Z}]}$ is the following:

$\star$	0	1	00	11	01	10
0	0	1	00	11	01	10
1	1	0	11	00	10	01
00	00	11	00	11	00	11
11	11	00	11	00	11	00
01	01	10	01	10	01	10
10	10	01	10	01	10	01

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00	00	11	00	11	00	11
11	11	00	11	00	11	00
01	01	10	01	10	01	10
10	10	01	10	01	10	01

Partitioning the universe into $\{0, 1\}$, $\{00, 11\}$ and $\{01, 10\}$ we obtain a congruence θ, such that $\mathbb{Z}_2^{[\mathcal{Z}]}/\theta$ is isomorphic to the left flip-flop monoid L_2^1.

Two-sided wreath product

Theorem

Let $(X, \backslash, /, \mathbf{S})$ consist of a set X together with a two-sided action of a semigroup $\mathbf{S}$ on X. Then the system of maps

$$\mathcal{S}(X, \mathbf{S}, X) = \big(\langle \lambda[a,b], \rho[a,b] \rangle \colon I[ab] \to I[a] \times I[b]\big),$$

where $I[s] = X$ for any $s \in S$, and

1. $\lambda[a,b] = b \backslash \_$ *for any* $a, b \in S$,
2. $\rho[a,b] = \_ / a$ *for all* $a, b \in S$.

is a $\lambda\rho$-system over $\mathbf{S}$. Moreover, for any semigroup $\mathbf{H}$, the $\lambda\rho$-product $\mathbf{H}^{[\mathcal{S}(X,\mathbf{S},X)]}$ is isomorphic to the two-sided wreath product of $\mathbf{H}$ by $\mathbf{S}$.

- Particular cases: one-sided wreath product, block product.

Preservation: monoids

Definition

Let P be a property of semigroups, and let $\mathcal{S} = (\mathbf{I}, \boldsymbol{\lambda}, \boldsymbol{\rho})$ be a $\lambda\rho$-system over $\mathbf{S}$. We say that $\mathcal{S}$ preserves P, if

$$\forall \mathbf{H} \colon P(\mathbf{H}) \Rightarrow P(\mathbf{H}^{[\mathcal{S}]}).$$

Theorem

Let $\mathcal{S} = (\mathbf{I}, \boldsymbol{\lambda}, \boldsymbol{\rho})$ be a $\lambda\rho$-system over $\mathbf{S}$. The following are equivalent:

1. *$\mathcal{S}$ is unit-preserving,*
2. *$\mathbf{S}$ is a monoid (with unit element 1) and the maps $\lambda[a,1]$ and $\rho[1,a]$ are the identity maps on $I[a]$, for each $a \in S$,*
3. *$\mathbf{S}$ is a monoid and there exists a nontrivial monoid $\mathbf{H}$ such that $\mathbf{H}^{[\mathcal{S}]}$ is a monoid.*

Preservation: equations and quasiequations

Theorem template

Let $\mathcal{S}$ be a $\lambda\rho$-system over **S**. *The following are equivalent:*

1. $\mathcal{S}$ *preserves foo.*
2. **S** *satisfies foo, and the maps $\lambda[a, b]$ and $\rho[b, a]$ satisfy foobar.*
3. **S** *satisfies foo and there exists some specific* **H** *satisfying foo such that $\mathbf{H}^{[\mathcal{S}]}$ satisfies foo.*

We know, for example, that foo can be:

- cancellativity (foobar: $\lambda[a, b]$ and $\rho[a, b]$ are surjective)
- idempotency (foobar: $\lambda[a, a] = \rho[a, a] = id$)
- commutativity (foobar: $\lambda[a, b] = \rho[b, a]$)
- medial identity: $xyzu = xzyu$
- left zero identity: $xy = x$

Preservation: groups

Theorem

Let $\mathcal{S}$ be a $\lambda\rho$-system over **S**. *The following are equivalent:*

1. *$\mathcal{S}$ preserves groups.*
2. **S** *is a group, the maps $\lambda[a, b]$ and $\rho[b, a]$ are bijective, for all $(a, b) \in S^2$, and for $b = 1$ they are identity maps.*
3. **S** *is a group and there exists a nontrivial group* **H** *such that $\mathbf{H}^{[\mathcal{S}]}$ is a group.*

- This is just like the template, but for groups we can get more.

Groups: bonus

Theorem

Let $\mathcal{S} = (\mathbf{I}, \boldsymbol{\lambda}, \boldsymbol{\rho})$ be a $\lambda\rho$-system over a semigroup $\mathbf{G}$. Then, the following are equivalent:

1. *$\mathcal{S}$ is group-preserving,*
2. *$\mathbf{G}$ is a group and $\mathcal{S}$ is unital,*
3. *$\mathbf{G}$ is a group and $(\mathbf{G}, \mathcal{S}) \cong (\mathbf{G}, \mathcal{S}(X, \mathbf{G}))$ with $\mathbf{G}$ acting on some set X.*
4. *$\mathbf{G}$ is a group and $\mathbf{G}^{[\mathcal{S}]}$ is isomorphic to a wreath product.*

- It shows that $\lambda\rho$-product is a reasonable generalisation of wreath product.

Semigroups: bonus

Recall:

Krohn-Rhodes Theorem

Every finite semigroup is a homomorphic image of a subsemigroup of an iterated wreath product of finite simple groups and the flip-flop monoid.

Semigroups: bonus

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Krohn-Rhodes Theorem

Every finite semigroup is a homomorphic image of a subsemigroup of an iterated wreath product of finite simple groups and the flip-flop monoid.

Since the flip-flop monoid can be decomposed as a $\lambda\rho$-product $\mathbb{Z}_2^{[\mathcal{Z}]}$ whose factors are $\mathbb{Z}_2$ and the two-element semilattice, we get:

Corollary

Every finite semigroup is a homomorphic image of a subsemigroup of an iterated $\lambda\rho$-product whose factors are finite simple groups and a two-element semilattice.

What next?

- Some categorical properties of $\lambda\rho$-products.
  - M. Botur, TK, "Beyond wreath and block", *Semigroup Forum*, forthcoming.
- More categorical properties, and some systematic handle on the preservation properties.
  - M. Botur, D. Lachman, TK, work very much in progress.
- Representations for some classes (varieties) of semigroups.
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- Applications for "algebras of logic".

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Thank you!