Don't be afraid of infinitary logics

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Logic4Peace

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Lindenbaum Lemma for certain infinitary consequence relations

Let $\vdash$ be a consequence relation on a countable set of formulas such that

- $\vdash$ has a countable axiomatization,
- $\mathrm{Th}(\vdash)$ is a frame,
- the intersection of any two finitely generated theories is finitely generated.

Then the finitely meet-irreducible theories form a basis of $\mathrm{Th}(\vdash)$.

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The "original" abstract Lindenbaum Lemma

Let $\vdash$ be a finitary consequence relation.
Then the meet-irreducible theories form a basis of $\mathrm{Th}(\vdash)$.

An example of infinitary many-valued logic

The standard MV-algebra $[\mathbf{0},\mathbf{1}]_{\text{Ł}}$ has the real unit interval $[0,1]$ as domain and operations $\to$, $\&$, $\vee$, and $\neg$ interpreted as:

$$x \to y = \min\{1, 1 - x + y\} \qquad x \,\&\, y = \max\{0, x + y - 1\}$$
$$x \vee y = \max\{x, y\} \qquad \neg x = 1 - x$$

The logic of standard MV-algebra (a.k.a. infintary Łukasiewicz logic):

$$\Gamma \models_{\mathrm{LSMVA}} \varphi \quad \text{iff} \quad (\forall e\colon \mathbf{Fm} \to [\mathbf{0},\mathbf{1}]_{\text{Ł}})(e[\Gamma] \subseteq \{1\} \implies e(\varphi) = 1)$$

Clearly, LSMVA is not finitary, e.g.:

$$\{\neg\varphi \to \varphi \,\&\, \overset{n}{\ldots} \,\&\, \varphi \mid n \geq 0\} \models_{\mathrm{LSMVA}} \varphi \qquad \text{but}$$
$$\{\neg\varphi \to \varphi \,\&\, \overset{n}{\ldots} \,\&\, \varphi \mid n \leq k\} \not\models_{\mathrm{LSMVA}} \varphi \qquad \text{for each } k$$

Two examples of infinitary modal logics

- In PDL:

$$\{[\alpha;\beta^n]\varphi \mid n \in \mathbb{N}\} \vDash [\alpha;\beta^*]\varphi$$

- In logics of common knowledge:

$$\{E^{n+1}\varphi \mid n \in \mathbb{N}\} \vDash C\varphi$$

A question and some answers

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A very incomplete list of existing answers:

1963 Hay: strongly complete axiomatization of Logic of Standard MV-Algebra

1977 Sundholm: strongly complete axiomatization of Von Wright's temporal logic

1993 Goldblatt: a general approach to modal logics with classical base

1994 Segerberg: a general method using saturated sets of formulas

2018 Bílková, Cintula, Lávička: a general method for certain algebraic logics

Consequence relations/logics

Fm: a countable set of formulas; in propositional case given by:

- a countable infinite set Var of propositional variables
- an at most countable propositional language $\mathcal{L}$

A consequence relation $\vdash$ is a relation between sets of formulas and formulas s.t.:

- $\{\varphi\} \vdash \varphi$ (Reflexivity)
- If $\Gamma \vdash \varphi$, then $\Gamma \cup \Delta \vdash \varphi$ (Monotonicity)
- If $\Gamma \vdash \varphi$ and $\Delta \vdash \psi$ for each $\psi \in \Gamma$, then $\Delta \vdash \varphi$ (Cut)

A consequence relation is

- finitary if: $\Gamma \vdash \varphi$ implies there is a finite $\Gamma' \subseteq \Gamma$ s.t. $\Gamma' \vdash \varphi$.
- structural (a.k.a. logic) if: $\Gamma \vdash \varphi$ implies $\sigma[\Gamma] \vdash \sigma(\varphi)$ for each substitution σ

Theories

$T \subseteq Fm$ is a theory of a CR $\vdash$ if whenever $T \vdash \varphi$, then $\varphi \in T$

A theory T is prime if it is not an intersection of two strictly bigger theories.

Abstract Lindenbaum lemma

Let $\vdash$ be a finitary CR. If $\Gamma \nvdash \varphi$, then there is a prime theory $T \supseteq \Gamma$ such that $\varphi \notin T$.

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Abstract Lindenbaum lemma

Let $\vdash$ be a finitary CR. If $\Gamma \nvdash \varphi$, then there is a prime theory $T \supseteq \Gamma$ such that $\varphi \notin T$.

The system of theories $\mathrm{Th}(\vdash)$ is a closure system

Lindenbaum lemma, then says that prime theories form its basis.

1st ingredient: Countable axiomatization

Proofs are trees labeled by formulas with no infinitely-long branch.

Let $Ł_\infty$ be the logic given by (instances) Łukasiewicz 4 axioms, *modus ponens*, and

$$\{\neg\varphi \to \varphi \,\&\, \overset{n}{\ldots} \,\&\, \varphi \mid n \geq 0\} \rhd \varphi \qquad \text{(Hay rule)}$$

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Not conversely: Clearly $Ł_\infty$ has a countable axiomatic system but as

$$\Gamma \vdash_{Ł_\infty} \varphi \qquad \text{implies} \qquad \Gamma \models_{\mathrm{LSMVA}} \varphi,$$

then $Ł_\infty$ is not finitary

2nd ingredient: Strong disjunction

A connective $\vee$ (primitive of defined) is called strong disjunction in $\vdash$ if:

$$\varphi \vdash \varphi \vee \psi \qquad \psi \vdash \varphi \vee \psi \tag{PD}$$

$$\frac{\Gamma \cup \Phi \vdash \chi \qquad \Gamma \cup \Psi \vdash \chi}{\Gamma \cup \{\varphi \vee \psi \mid \varphi \in \Phi, \psi \in \Psi\} \vdash \chi} \tag{sPCP}$$

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If $\vee$ is a strong disjunction, then a theory T is prime iff for each φ and ψ:

if $\varphi \vee \psi \in T$, then $\varphi \in T$ or $\psi \in T$.

The main result for logics

Lindenbaum Lemma for certain infinitary logics

Let $\vdash$ be a logic with a countable axiomatization and a strong disjunction.
If $\Gamma \nvdash \varphi$, then there is a prime theory $T \supseteq \Gamma$ such that $\varphi \notin T$.

The need for countable axiomatization

Consider language with $\vee$, and a constant $\mathbf{i}$ for each $i \in \omega$.

Let $\vdash$ be the expansion of the disjunction-fragment of classical logic by:

$$\{\mathbf{i} \vee \chi \mid i \in C\} \rhd \chi$$

for each infinite set $C \subseteq \omega$.

Then $\vee$ is a strong disjunction in $\vdash$ but the Lindenbaum Lemma fails; indeed for $\Gamma = \{\mathbf{2i} \vee \mathbf{2i+1} \mid i \in \omega\}$ and each prime $T \subseteq \Gamma$ we have:

$$\Gamma \nvdash \mathbf{0} \qquad\qquad T \vdash \mathbf{0}$$

The need for strong disjunction

Consider the logic $\vdash$ with unary operation $\Box$ given by rules (for $n \geq 0$):

$$\{\Box^m \varphi \mid m > n\} \rhd \varphi$$

Clearly $\vdash$ has a countable axiomatization and

$$\Gamma \cup \{\varphi\} \vdash \chi \qquad \text{iff} \qquad \chi = \varphi \text{ or } \Gamma \vdash \chi$$

Thus if T is a theory, so is $T \cup \{\psi\}$ and so only Fm is a prime theory

As there are non-trivial theories (i.e., $\emptyset$), Lindenbaum lemma has to fail

A characterization of strong disjunction

Theorem

Let $\vdash$ be a logic with axiomatization $\mathcal{AS}$. Then $\vee$ is a strong disjunction iff

$$\varphi \vdash \varphi \vee \psi \qquad \varphi \vee \psi \vdash \psi \vee \varphi \qquad \varphi \vee \varphi \vdash \varphi$$

$$\{\gamma \vee \chi \mid \gamma \in \Gamma\} \vdash \varphi \vee \chi \qquad \textit{for each } \Gamma \rhd \varphi \textit{ from } \mathcal{AS}$$

We can easily show that:

$$\{(\neg\varphi \to \varphi \,\&\, \overset{n}{\ldots} \,\&\, \varphi) \vee \chi \mid n \geq 0\} \vdash_{Ł_\infty} \varphi \vee \chi,$$

and so $Ł_\infty$ has a strong disjunction

An application: L_∞ is the logic of standard MV-algebra

1) We know that it has a countable axiomatization and $\vee$ is its strong disjunction

2) Thus if $\Gamma \nvdash_{\mathrm{L}_\infty} \varphi$, there is a prime theory $T \supseteq \Gamma$ st. $\varphi \notin T$

3) Take the Lindenbaum–Tarski algebra of T: we know it is a simple MV-chain (thanks to the Hay rule)

4) Each simple MV-chain is embeddable into standard MV-algebra $[\mathbf{0},\mathbf{1}]_{Ł}$

5) Thus we have a $[\mathbf{0},\mathbf{1}]_{Ł}$-evaluation e such that $e(\Gamma) \subseteq \{\overline{1}\}$ but $e(\varphi) < 1$

Two kinds of disjunction

A connective $\vee$ (primitive of defined) is called strong disjunction in $\vdash$ if:

$$\varphi \vdash \varphi \vee \psi \qquad \psi \vdash \varphi \vee \psi \tag{PD}$$

$$\frac{\Gamma \cup \Phi \vdash \chi \qquad \Gamma \cup \Psi \vdash \chi}{\Gamma \cup \{\varphi \vee \psi \mid \varphi \in \Phi, \psi \in \Psi\} \vdash \chi} \tag{sPCP}$$

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$$\varphi \vdash \varphi \vee \psi \qquad \psi \vdash \varphi \vee \psi \tag{PD}$$

$$\frac{\Gamma \cup \{\varphi\} \vdash \chi \qquad \Gamma \cup \{\psi\} \vdash \chi}{\Gamma \cup \{\varphi \vee \psi\} \vdash \chi} \tag{PCP}$$

In a finitary logic each disjunction is strong but not vice-versa

All that disjoints is not a disjunction

The lattice connective $\vee$ need not satisfy PCP, e.g. in global S4

Indeed it would entail that $\varphi \vee \neg\varphi \vdash^{g}_{\mathrm{S4}} \Box\varphi \vee \neg\varphi$, i.e.,

$$\vdash^{g}_{\mathrm{S4}} \varphi \to \Box\varphi$$

On the other hand we can show that $\Box\varphi \vee \Box\psi$ is a (strong) disjunction in $\mathrm{S4}$:

$$\frac{\Gamma \cup \{\varphi\} \vdash^{g}_{\mathrm{S4}} \chi \qquad \Gamma \cup \{\psi\} \vdash^{g}_{\mathrm{S4}} \chi}{\Gamma \cup \{\Box\varphi \vee \Box\psi\} \vdash^{g}_{\mathrm{S4}} \chi}$$

How to get rid of structurality?

Let $\vdash$ be a logic with a disjuntion $\vee$. Then

- the intersection of any two finitely generated theories is finitely generated; in particular for each finite $\Phi, \Psi \subseteq Fm$:

$$\mathrm{Th}_{\vdash}(\Phi) \cap \mathrm{Th}_{\vdash}(\Psi) = \mathrm{Th}_{\vdash}(\{\varphi \vee \psi \mid \varphi \in \Phi, \psi \in \Psi\})$$

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- $\vee$ is a strong disjunction iff $\mathrm{Th}(\vdash)$ is a frame, i.e., for each $\mathcal{S} \cup \{T\} \subseteq \mathrm{Th}(\vdash)$:

$$T \cap \bigvee_{S \in \mathcal{S}} S = \bigvee_{S \in \mathcal{S}} (T \cap S).$$

A more general result

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If $\Gamma \nvdash \varphi$, then there is a prime theory $T \supseteq \Gamma$ such that $\varphi \notin T$.

A more general result

Lindenbaum Lemma for certain infinitary CRs

Let $\vdash$ be a CR with a countable axiomatization such that $\mathrm{Th}(\vdash)$ is a frame and the intersection of any two finitely generated theories is finitely generated.

If $\Gamma \nvdash \varphi$, then there is a prime theory $T \supseteq \Gamma$ such that $\varphi \notin T$.