Tractable depth-bounded approximations to First-Degree Entailment (**FDE**)

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Logic4Peace
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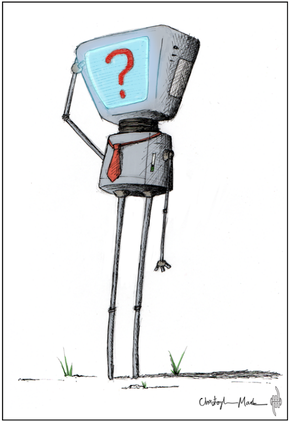
Plan for the talk

1. Introduction
   - Motivation
   - The "depth-bounded" approach
   - First-Degree Entailment (**FDE**)
2. Depth-bounded **FDE**
   - Applying the "depth-bounded" approach
   - 0-depth consequence
   - *k*-depth consequence
3. Final remarks

- Many interesting propositional logics are likely to be **intractable**.
  - **CPL** and **FDE** are co-NP complete.
  - **IPL** is PSPACE-complete.
- Difficulties in areas that need less idealized models of rationality and computation.
  - Economics, AI, Cognitive Science, Philosophy, etc.
- Tractable approximations to **CPL** have been investigated since the 1990's (Cadoli & Schaerf, Finger & Wasserman, Massacci, Stålmarck, Crawford & Etherington, Lakemeyer & Levesque).



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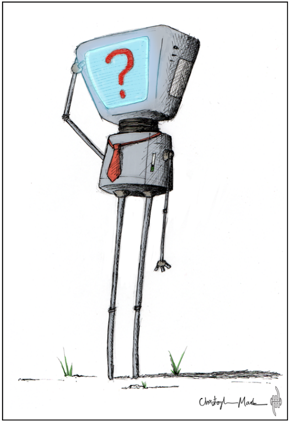

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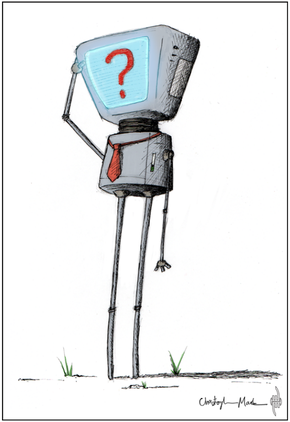

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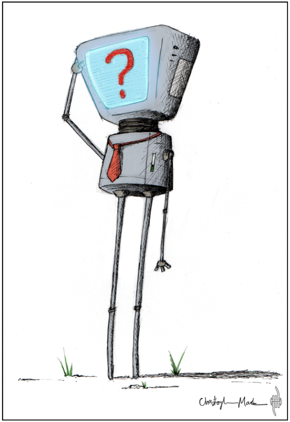

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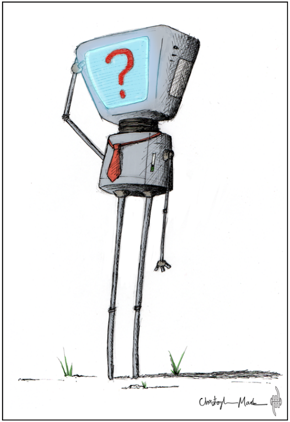

- A more recent development is the "depth-bounded" approach (D'Agostino et al., 2009, 2013, D'Agostino, 2015).
- Based on the distinction between **actual** and **virtual** information.
- Admits of a 3-valued non-deterministic semantics (see Avron & Zamansky, 2011), whose values have a natural informational interpretation, and a non-standard proof-theoretical characterization.
- Leads to defining a hierarchy of **tractable** approximations to **CPL**, in terms of the maximum number of allowed nested applications of a single branching structural rule which expresses the Principle of Bivalence.
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Standard values

- Put forward as the logic in which "a computer **should** think", and admits of an intuitive semantics based on informational values (Dunn, 1976; Belnap, 1977).
- 4 possible ways in which an atom p can belong to the present state of information of a computer's database, in turn fed by a set of sources:
  - **t**: the computer is told that p is true by some source, without being told that p is false by any source;
  - **f**: it's told that p is false but never told that p is true;
  - **b**: it's told that p is true by some source and that p is false by some other source (or the same at different moments);
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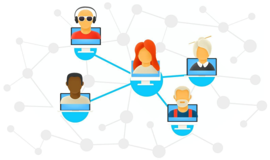

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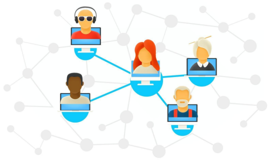

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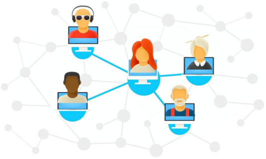

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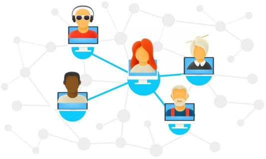

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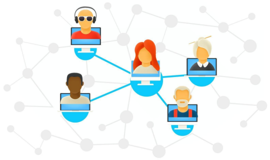

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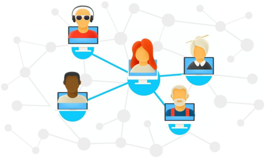

Truth-tables and consequence

$\widetilde{\vee}$	t	f	b	n
t	t	t	t	t
f	t	f	b	n
b	t	b	b	t
n	t	n	t	n

$\widetilde{\wedge}$	t	f	b	n
t	t	f	b	n
f	f	f	f	f
b	b	f	b	f
n	n	f	f	n

$\widetilde{\neg}$	
t	f
f	t
b	b
n	n

Definition

A **4**-valuation is a function $v : F(\mathcal{L}) \longrightarrow \mathbf{4}$, that agrees with the tables.

Definition

$\Gamma \vDash_{\mathsf{FDE}} A$ iff for every **4**-valuation v, if $v(B) \in \{\mathbf{t}, \mathbf{b}\}$ for all $B \in \Gamma$, then $v(A) \in \{\mathbf{t}, \mathbf{b}\}$.

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f	**t**	**f**	**b**	**n**
b	**t**	**b**	**b**	**t**
n	**t**	**n**	**t**	**n**

$\widetilde{\wedge}$	**t**	**f**	**b**	**n**
t	**t**	**f**	**b**	**n**
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First key observation: the need of imprecise values

- Despite its informational flavour, **FDE** is co-NP complete (see Urquhart, 1990; Arieli & Denecker, 2003), and so an idealized model of how an agent **can** think.
- Except for **b**, the standard values cannot be taken as **stable** without assuming complete information about the set of sources:
  - **b**: **there is** at least a source assenting to p and at least a source dissenting from p;
  - **t**, **f** and **n**: **there is no** source such that...
- What if the agent does not have such a complete knowledge about the sources (e.g., the set of sources is "open")?
- This motivates the need for a stable **imprecise** value such as "**t** or **b**", implicit in the choice of designated values in the semantics of **FDE**.

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Strategy

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  - T A: x holds that A is at least true, $v(A) \in \{\mathbf{t}, \mathbf{b}\}$;
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Second key observation

- No reason to assume that an agent is "told" about the values of **atoms only**.
- Agents may be told that a disjunction is true without being told which of the two disjuncts is the true one, and dually for conjunctions.
- For example, being told that Alice and Bob are siblings (either they have the same mother or they have the same father).
- The value of an atom may be completely **undefined** when the agent's information is insufficient to even establish any of the imprecise values.

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Linear introduction rules

$$\frac{\mathsf{F}\,A}{\mathsf{F}\,A \wedge B} \qquad \frac{\mathsf{F}\,B}{\mathsf{F}\,A \wedge B} \qquad \frac{\mathsf{F}^*\,A}{\mathsf{F}^*\,A \wedge B} \qquad \frac{\mathsf{F}^*\,B}{\mathsf{F}^*\,A \wedge B}$$

$$\frac{\mathsf{T}\,A}{\mathsf{T}\,A \vee B} \qquad \frac{\mathsf{T}\,B}{\mathsf{T}\,A \vee B} \qquad \frac{\mathsf{T}^*\,A}{\mathsf{T}^*\,A \vee B} \qquad \frac{\mathsf{T}^*\,B}{\mathsf{T}^*\,A \vee B}$$

$$\frac{\begin{array}{c}\mathsf{T}\,A\\ \mathsf{T}\,B\end{array}}{\mathsf{T}\,A \wedge B} \qquad \frac{\begin{array}{c}\mathsf{F}\,A\\ \mathsf{F}\,B\end{array}}{\mathsf{F}\,A \vee B} \qquad \frac{\begin{array}{c}\mathsf{T}^*\,A\\ \mathsf{T}^*\,B\end{array}}{\mathsf{T}^*\,A \wedge B} \qquad \frac{\begin{array}{c}\mathsf{F}^*\,A\\ \mathsf{F}^*\,B\end{array}}{\mathsf{F}^*\,A \vee B}$$

$$\frac{\mathsf{T}\,A}{\mathsf{F}^*\,\neg A} \qquad \frac{\mathsf{F}\,A}{\mathsf{T}^*\,\neg A} \qquad \frac{\mathsf{T}^*\,A}{\mathsf{F}\,\neg A} \qquad \frac{\mathsf{F}^*\,A}{\mathsf{T}\,\neg A}$$

Linear elimination rules

$$\frac{\begin{array}{c}\mathsf{F}\,A \wedge B\\ \mathsf{T}\,A\end{array}}{\mathsf{F}\,B} \qquad \frac{\begin{array}{c}\mathsf{F}\,A \wedge B\\ \mathsf{T}\,B\end{array}}{\mathsf{F}\,A} \qquad \frac{\begin{array}{c}\mathsf{F}^*\,A \wedge B\\ \mathsf{T}^*\,A\end{array}}{\mathsf{F}^*\,B} \qquad \frac{\begin{array}{c}\mathsf{F}^*\,A \wedge B\\ \mathsf{T}^*\,B\end{array}}{\mathsf{F}^*\,A}$$

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$$\frac{\mathsf{T}\,\neg A}{\mathsf{F}^*\,A} \qquad \frac{\mathsf{F}\,\neg A}{\mathsf{T}^*\,A} \qquad \frac{\mathsf{T}^*\,\neg A}{\mathsf{F}\,A} \qquad \frac{\mathsf{F}^*\,\neg A}{\mathsf{T}\,A}$$

Intelim sequences and tractability

$$\mathsf{T}\,\neg(A \vee B)^{@}$$
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Branching structural rules and virtual information

- The intelim rules are not complete for full **FDE**. Completeness is obtained by adding only:

$$PB: \quad \frac{}{\mathsf{T}\,A \mid \mathsf{F}\,A} \qquad PB^*: \quad \frac{}{\mathsf{T}^*\,A \mid \mathsf{F}^*\,A}$$

- One of the two cases must obtain considering the whole set of sources even if the agent has no actual information about which is the case.
- We call the information expressed by each of the two complementary signed formulae "**virtual**".
- The more virtual information needs to be invoked via PB or PB^*, the harder the inference is.
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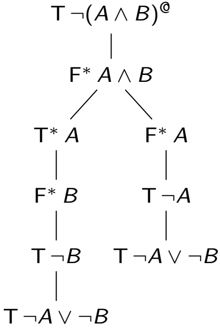
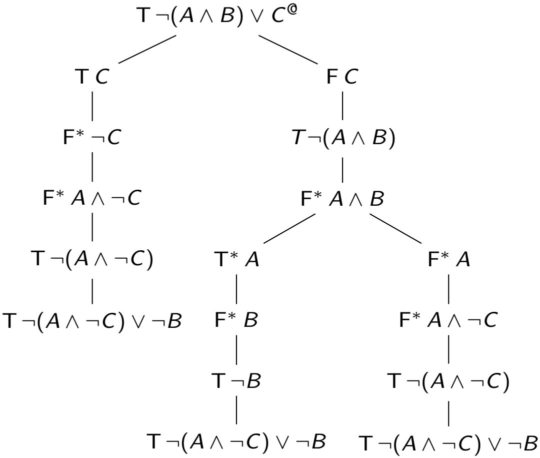


k=1



k=2

Tractability and non-deterministic semantics

- Leads to an infinite hierarchy of tractable depth-bounded approximations, in terms of the maximum number of nested applications of *PB* and PB^* that are allowed.
- Each k-depth consequence relation, $k \geq 0$, can be decided in time $\boldsymbol{O}(\boldsymbol{n}^{k+2})$.
- Admits of a 5-valued non-deterministic semantics (see Avron & Zamansky, 2011): takes the **signs as imprecise values**, and adds a fifth value standing for the case where the value of a formula is completely **undefined** in that the information is insufficient to even establish any of the imprecise values.

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- The method easily extends to **LP** and $\mathbf{K}_3$.
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Virtual information in CPL

$$\frac{p \quad p \to q \quad q \to r \quad r \to s \quad s \to t \quad t \to u \quad u \to v \quad v \to w \quad w \to x \quad x \to y \quad y \to z}{z} \quad < \quad \frac{p \lor q \quad p \to r \quad q \to r}{r} \quad < \quad \frac{p \lor q \lor r \quad p \lor q \lor \neg r \quad p \lor \neg q \lor s \quad p \lor \neg q \lor \neg s \quad \neg p \lor q \lor t \quad \neg p \lor q \lor \neg t \quad \neg p \lor \neg q \lor u \quad \neg p \lor \neg q \lor \neg u}{\curlywedge}$$

no virtual info | virtual info | nested virtual info

1-depth intelim refutation in FDE

$$\mathsf{T}\, A \vee (B \wedge C)^{@}$$

$$\mathsf{F}\, (A \vee B) \wedge (A \vee C)^{@}$$

$\mathsf{T}\, A$	$\mathsf{F}\, A$
$\mathsf{T}\, A \vee B$	$\mathsf{T}\, B \wedge C$
$\mathsf{F}\, A \vee C$	$\mathsf{T}\, B$
$\mathsf{F}\, A$	$\mathsf{T}\, C$
	$\mathsf{T}\, A \vee B$
	$\mathsf{T}\, A \vee C$
	$\mathsf{T}\, (A \vee B) \wedge (A \vee C)$

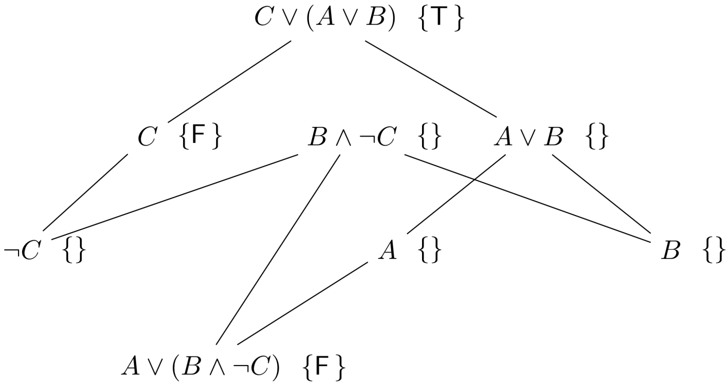

Figure 1: Initialized graph

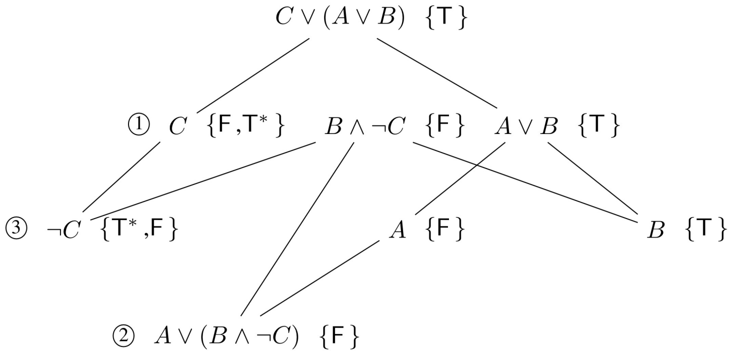

Figure 2: Saturated graph

5N-tables for **FDE**

$\widetilde{\vee}$	t	f	t^*	f^*	$\bot$
t	$\{t\}$	$\{t\}$	$\{\mathbf{t}\}$	$\{t\}$	$\{t\}$
f	$\{t\}$	$\{f\}$	$\{t^*\}$	$\{\bot\}$	$\{\bot, t^*\}$
t^*	$\{\mathbf{t}\}$	$\{t^*\}$	$\{t^*\}$	$\{t^*\}$	$\{t^*\}$
f^*	$\{t\}$	$\{\bot\}$	$\{t^*\}$	$\{f^*\}$	$\{\bot, t\}$
$\bot$	$\{t\}$	$\{\bot, t^*\}$	$\{t^*\}$	$\{\bot, t\}$	$\{t, t^*, \bot\}$

$\widetilde{\wedge}$	t	f	t^*	f^*	$\bot$
t	$\{t\}$	$\{f\}$	$\{\bot\}$	$\{f^*\}$	$\{\bot, f^*\}$
f	$\{f\}$	$\{f\}$	$\{f\}$	$\{\mathbf{f}\}$	$\{f\}$
t^*	$\{\bot\}$	$\{f\}$	$\{t^*\}$	$\{f^*\}$	$\{\bot, f\}$
f^*	$\{f^*\}$	$\{\mathbf{f}\}$	$\{f^*\}$	$\{f^*\}$	$\{f^*\}$
$\bot$	$\{\bot, f^*\}$	$\{f\}$	$\{\bot, f\}$	$\{f^*\}$	$\{f, f^*, \bot\}$

$\widetilde{\neg}$	
t	f^*
f	t^*
t^*	f
f^*	t
$\bot$	$\bot$

LP/K$_3$ standard tables

$\widetilde{\vee}$	true	false	i
true	true	true	true
false	true	false	i
i	true	i	i

$\widetilde{\wedge}$	true	false	i
true	true	false	i
false	false	false	false
i	i	false	i

$\widetilde{\neg}$	
true	false
false	true
i	i

$\widetilde{\rightarrow}$	true	false	i
true	true	false	i
false	true	true	true
i	true	i	i

Additional intelim rules for **LP** and $\mathbf{K}_3$

$$\frac{\mathsf{F}^* A}{\mathsf{T}\, A \to B} \qquad \frac{\mathsf{T}\, B}{\mathsf{T}\, A \to B} \qquad \frac{\mathsf{T}^* A \quad \mathsf{F}\, B}{\mathsf{F}\, A \to B}$$

$$\frac{\mathsf{F}\, A}{\mathsf{T}^* A \to B} \qquad \frac{\mathsf{T}^* B}{\mathsf{T}^* A \to B} \qquad \frac{\mathsf{T}\, A \quad \mathsf{F}^* B}{\mathsf{F}^* A \to B}$$

$$\frac{\mathsf{F}\, A \to B}{\mathsf{T}^* A} \qquad \frac{\mathsf{F}\, A \to B}{\mathsf{F}\, B} \qquad \frac{\mathsf{F}^* A \to B}{\mathsf{T}\, A} \qquad \frac{\mathsf{F}^* A \to B}{\mathsf{F}^* B}$$

$$\frac{\mathsf{T}\, A \to B \quad \mathsf{T}^* A}{\mathsf{T}\, B} \qquad \frac{\mathsf{T}^* A \to B \quad \mathsf{T}\, A}{\mathsf{T}^* B} \qquad \frac{\mathsf{T}\, A \to B \quad \mathsf{F}\, B}{\mathsf{F}^* A} \qquad \frac{\mathsf{T}^* A \to B \quad \mathsf{F}^* B}{\mathsf{F}\, A}$$

$$\frac{\mathsf{T}^* A}{\mathsf{T}\, A} \qquad \frac{\mathsf{F}\, A}{\mathsf{F}^* A} \qquad\qquad \frac{\mathsf{T}\, A}{\mathsf{T}^* A} \qquad \frac{\mathsf{F}^* A}{\mathsf{F}\, A}$$