# VALIDITIES FOR RESIDUATED ALGEBRAS OF BINARY RELATIONS

SZABOLCS MIKULÁS

We will look at algebras of binary relations whose signatures $\Lambda$ contain relation composition ; and its residuals $\setminus$ (right) and $/$ (left). We will also assume that an ordering $\leq$ is available, either as a primitive relation symbol, or defined by using the (semi)lattice operations join $+$ or meet $\cdot$ .

Terms are interpreted in an algebra $\mathfrak{C}$ with base $U_{\mathfrak{C}}$ in the usual manner: join $+$ is union, meet $\cdot$ is intersection and

$$x\,;y = \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : (u,w) \in x \text{ and } (w,v) \in y \text{ for some } w\}$$
$$x \setminus y = \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : \text{for every } w, (w,u) \in x \text{ implies } (w,v) \in y\}$$
$$x \,/\, y = \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : \text{for every } w, (v,w) \in y \text{ implies } (u,w) \in x\}$$

We will also need the identity constant $1'$ interpreted as

$$1' = \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : u = v\}$$

We will look at two notions of semantics. Let $\tau, \sigma$ be two terms. We say that the (in)equality $\tau \leq \sigma$ is (standard) valid, in symbols $\models \tau \leq \sigma$, if the interpretation of $\tau$ is a subset of the interpretation of $\sigma$ in every algebra. On the other hand, state-semantics is defined for terms. We say that $\tau$ is state-valid, in symbols $\models_s \tau$, if $1' \leq \tau$ is (standard) valid. These semantics can be restricted to special classes of algebras. In particular, we will look at commutative algebras, where $x\,;y = y\,;x$ is valid. The corresponding notion of validity is denoted by using a superscript: $\models^c$ and $\models^c_s$.

We will consider signatures $\{;, \setminus, /\} \subseteq \Lambda \subseteq \{\cdot, +, ;, \setminus, /\}$ and investigate when validities $\models$, $\models^c$ and state-validities $\models_s$, $\models^c_s$ are finitely axiomatizable. That is, we look for finite set of axioms and derivation rules such that all (state-)validities can be derived. We will see that, in this respect, lower-semilattice ordered algebras generally behave better than upper-semilattice ordered algebras, although some problems are still open.

Department of Computer Science and Information Systems, Birkbeck, University of London