SUBCOMPLETIONS OF ATOMIC REPRESENTABLE RELATION ALGEBRAS

ROGER D. MADDUX

Extended Abstract. Hodkinson [1] solved a problem of Monk [2] by constructing an atomic representable relation algebra $\mathbf{B} \in \mathsf{RRA}$ whose completion $\mathbf{C}$ is not representable. Since RRA is a variety and $\mathbf{C}$ is not in RRA, there must be an equation ϵ that holds in RRA but fails in $\mathbf{C}$. The subalgebra $\mathbf{A}$ of $\mathbf{C}$ generated by the finitely many values assigned to the variables occurring in ϵ is not representable because it too fails to satisfy ϵ. $\mathbf{A}$ is an example of a finitely-generated relation algebra that is a subalgebra of the non-representable completion of an atomic RRA. What relation algebras can occur as $\mathbf{A}$? We show that every finite Monk algebra with six or more colors is such an algebra.

A **Monk algebra** is any atomic symmetric integral relation algebra $\mathbf{A}$ obtained by splitting from some $\mathbf{E}_q^{23}$ with $4 \leq q \in \omega$. $\mathbf{E}_q^{23}$ is the finite symmetric integral relation algebra with q atoms such that if a, b are distinct diversity atoms then $a ; b = 0$' and $a ; a = \overline{a}$. $\mathbf{A}$ is obtained from $\mathbf{E}_q^{23}$ by **splitting** if $\mathbf{E}_q^{23} \subseteq \mathbf{A}$, every atom x of $\mathbf{A}$ is contained in an atom $\mathsf{c}(x)$ of $\mathbf{E}_q^{23}$, and for all atoms x, y of $\mathbf{A}$, if $x, y \leq 0$' then

$$x ; y = \begin{cases} \mathsf{c}(x) ; \mathsf{c}(y) \cdot 0\text{'} & \text{if } x \neq y \\ \mathsf{c}(x) ; \mathsf{c}(y) & \text{if } x = y. \end{cases}$$

The $q - 1$ diversity atoms of the subalgebra $\mathbf{E}_q^{23} \subseteq \mathbf{A}$ are called the **colors** of $\mathbf{A}$.

From an arbitrary finite symmetric integral relation algebra $\mathbf{A}$ and its subalgebra $\mathbf{E}$ we construct a complete atomic algebra $C_{\mathbf{E}}(\mathbf{A})$ and let $\mathbf{B}$ be the subalgebra of $C_{\mathbf{E}}(\mathbf{A})$ generated by the atoms of $C_{\mathbf{E}}(\mathbf{A})$. This construction and its properties are the main contribution of this paper. In particular, every finitely-generated subalgebra of $C_{\mathbf{E}}(\mathbf{A})$ is *finite*. When $7 \leq q < \omega$ and $\mathbf{A}$ is a Monk algebra obtained from $\mathbf{E}_q^{23}$, there is a subalgebra $\mathbf{E} \subseteq \mathbf{E}_q^{23} \subseteq \mathbf{A}$ such that

- $\mathbf{B}$ is a countable, atomic, symmetric, integral relation algebra generated by its atoms,
- $C_{\mathbf{E}}(\mathbf{A})$ and $\mathbf{B}$ have the same atom structure,
- $C_{\mathbf{E}}(\mathbf{A})$ is isomorphic to the complex algebra of the atom structure of $\mathbf{B}$,
- $C_{\mathbf{E}}(\mathbf{A})$ is the completion of $\mathbf{B}$,
- there is a subalgebra $\mathbf{A}' \subseteq C_{\mathbf{E}}(\mathbf{A})$ with $\mathbf{A}' \cong \mathbf{A}$,
- every finitely generated subalgebra of $\mathbf{B}$ is finite,
- $\mathbf{B}$ is representable,
- $\mathbf{B}$ is not completely representable,
- $C_{\mathbf{E}}(\mathbf{A})$ is not representable,
- $C_{\mathbf{E}}(\mathbf{A})$ is isomorphic to the relation algebraic reduct of a complete atomic q-dimensional cylindric algebra $\mathbf{Ca}\,(B_q(C_{\mathbf{E}}(\mathbf{A}))) \in \mathsf{CA}_q$ such that $\mathbf{Ca}\,(B_q(C_{\mathbf{E}}(\mathbf{A}))) \notin \mathsf{SNr}_q\mathsf{CA}_{q+1}$.

Thus the atom-generated subalgebra $\mathbf{B}$ of the complete atomic relation algebra $C_{\mathbf{E}}(\mathbf{A})$ is an atomic symmetric integral representable relation algebra with finite finitely-generated subalgebras whose completion $C_{\mathbf{E}}(\mathbf{A})$ contains a copy of the Monk algebra $\mathbf{A}$.

References

1. Ian Hodkinson, *Atom structures of cylindric algebras and relation algebras*, Ann. Pure Appl. Logic **89** (1997), no. 2-3, 117–148. MR 1490103 (99c:03103)
2. J. Donald Monk, *Completions of Boolean algebras with operators*, Math. Nachr. **46** (1970), 47–55. MR 0277369 (43 #3102)

Department of Mathematics, 396 Carver Hall, Iowa State University, Ames, Iowa 50011-2066, USA
E-mail address: maddux@iastate.edu

Date: June 15, 2012.