Public Announcement Logics with Constrained Protocols

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Abstract

Public announcement logic (PAL) is a paradigm case of dynamic epistemic logic, which models how agents' epistemic states change when pieces of information are communicated publicly. PAL extends epistemic logic with the operator $[A]$, where the intended reading of $[A]\phi$ is "After a public announcement that A, ϕ holds." This logic has recently received two improvements. One improvement, studied in [1], is to extend PAL with a generalized public announcement operator that allows quantification over public announcements. The other, studied in [5, 6], is a semantic setting to model "announcement protocols" to restrict the announcable sequences of formulas, while whatever is true is assumed to be announcable in PAL itself. The purpose of the present paper is to merge these two kinds of improvements. We consider the extension of public announcement logic with the generalized public announcement operator in the semantic setting of restricted announcement protocols.

1 Introduction

Dynamic epistemic logic models agents' epistemic shifts through informational updates. One paradigm is public announcement logic (PAL) (see e.g. [16, 12, 8]), which extends epistemic logic with the operator $[A]$ (for every formula A), where the intended reading of $[A]\phi$ is "After the public announcement that A, ϕ holds." The semantics of the operator is given, in a usual framework of epistemic logic, by:

$$\mathcal{M}, w \models [A]\phi \text{ iff } \mathcal{M}, w \models A \text{ implies } \mathcal{M}|_A, w \models \phi,$$

where $\mathcal{M}|_A$ is the model obtained by restricting $\mathcal{M}$ to the set of points at which A is true. As such, the PAL-operator $[A]$ can capture the informational events beyond public announcements that eliminate the epistemic possibility of non-A, and can model a variety of notions such as "after learning A," "after observing A," etc. Thus, the logical setting of PAL finds a wide range of applications and highlights various aspects of knowledge and communication.

Upon the original framework of PAL, two improvements have been recently given. One concerns the quantification over public announcements or whatever events PAL captures. There seems to be various epistemic concepts that, implicitly or explicitly, involves such quantification, and it is very useful to have an operator that does the quantification to model those concepts. A prime example is the notion of knowability, about which the well-known Fitch's paradox ([10])−i.e. if there is an unknown truth that p, then it is unknowable that p is an unknown truth−has drawn a wide attention in the field of epistemology. When we introduce the operator $\Diamond$ in the PAL-setting, where the intended reading of and $\Diamond\phi$ is "There is some public announcement after which ϕ holds," the formula $\Diamond K\phi$ would mean "ϕ is knowable." The extension of PAL with this generalized PAL-operator is called *aribitrary public announcement logic* ($APAL$) and studied in [1].

However, the original PAL-setting is still very much limited to model such interesting concepts in more realistic situations. For example, let us observe the following modelling problem concerning $APAL$. Take a case of the *Muddy Children puzzle* (MC) (see e.g. [9]) with three children 1, 2, 3. Let all 1, 2, and 3 be dirty. Let us put d_i as "i is dirty" for $i = 1, 2, 3$. First, it is a routine task to encode into an epistemic model all the epistemic situations of the children at the start (i.e. every child knows whether every other child is dirty, but not whether he himself is.). The children's answers at each round of father's question ("Does anybody know if he is dirty?") as well as the father's initial announcement ("At least one of you is dirty") are also interpretable by the PAL-operators as the events that eliminate certain epistemic possibilities. Then, consider, say, the question whether the dirty child 1 can know in the first question round that he is dirty. One might first try to capture this question by approximating it by $\langle\bigvee_{i=1}^{3} d_i\rangle\Diamond K_1 d_1$ in the $APAL$-setting (The first announcement operator corresponds to the father's initial announcement.). However, this sentence does not reflect the correct solution of MC. For we know by the solution that, until the second round finishes, 1 cannot know that he is dirty, while the $APAL$-formula will be true given that we would have, for instance, $\langle d_1\rangle K_1 d_1$ true. This problem is due to the fact that, whereas whatever is true is announcable in the original PAL-setting, saying publicly that d_1 is not allowed in the conversational constraints of MC.

Here, the other recent improvement on the original PAL-setting would give us a hand. The improvement challenges the above assumption of PAL (or the assumption of dynamic epistemic logic in general) that an informational event can take place whenever its precondition is satisfied. This assumption is evident in the above truth definition for the PAL-operator, since the only factor concerning A's announcability is its truth in the model. However, this seems too idealistic, since there are often various constraints, beyond the truth of preconditions, concerning whether a given informational event can happen, as we observed in the above MC example. These considerations motivate a semantic setting that models "PAL-protocols" to restrict the announcable sequences of formulas. This can be done by merging the PAL-setting with epistemic temporal logic (e.g. [9, 15]). In [5, 6], the logic of the PAL-operators with the kind of semantic setting, called $TPAL$, has been studied.

Having the development of $TPAL$, we seek the system that extends $TPAL$

with the generalized PAL-operator. For the problem about the lack of constraints on the availability of informational events beyond truth is much more evident when the system is with the generalized PAL-operator, as we observed in the above MC example, than when we restrict ourselves only to the specific announcements by the usual PAL-operators. (In the latter case, we are supposed to have specific announcements of interest in our subject of study, about which we would like to analyze what the effects are, and it is enough if we carefully do not put the statements that are not available in the situation.) Therefore, the purpose of the present paper is to consider the generalized PAL-operator in the $TPAL$ setting.

We proceed as follows. In Section 2, we will start out by reviewing PAL and $TPAL$ as in [6]. In Section 3, we will introduce the two generalized PAL-operators $\Diamond$ and $\Diamond^*$ to make a distinction between single announcements and sequences of announcements, which is necessary in the $TPAL$-setting. We then see their semantic properties. In Section 4, we axiomatize the logics with $\Diamond$ and $\Diamond^*$. The soundness and completeness proofs will be given in a Henkin-style argument. In Section 5, we present on a wider perspective given by product update with event models to conclude the paper.

2 $TPAL$ with Relativized PAL-Protocols

2.1 PAL

Let us start out by reviewing the system of PAL. Fix N as a finite set of agents and P, as a countable set of propositional letters.

Definition 2.1 (Language)**.** The language $\mathcal{L}_{PAL}$ of PAL is inductively defined as follows:

$$\phi \ ::= p \mid \top \mid \neg\phi \mid \phi \wedge \phi \mid K_i\phi \mid [\phi]\phi$$

where $p \in P$ and $i \in N$. Also the language $\mathcal{L}_{el}$ of multi-agent epistemic logic is the fragment of $\mathcal{L}_{PAL}$ without $[\phi]\phi$.

The intended readings of $K_i\phi$ and $[\psi]\phi$ are respectively "an agent i knows that ϕ" and "after the public announcement that ψ, ϕ." The duals of K_a and $[\phi]$ are denoted by $\langle i\rangle$ and $\langle\phi\rangle$, where the intended readings of $\langle i\rangle\phi$ and $\langle\psi\rangle\phi$ are "an agent i considers ϕ possible" and "the public announcement that ψ can be made after which ϕ." The other boolean connectives are defined in a familiar way.

Definition 2.2 (Truth)**.** Let $\mathcal{M}$ be an epistemic model $(W, (\sim_i)_{i\in N}, V)$, where W is a nonempty set, $\sim_i$, an equivalence relation on W, and $V : P \to \wp(W)$, a valuation function on P. Let $w \in W$. The truth of $\phi \in \mathcal{L}_{PAL}$ at w in $\mathcal{M}$ is inductively defined as follows:

$$
\begin{array}{lll}
\mathcal{M}, w \models p & \text{iff} & w \in V(p) \quad (\text{with } p \in P) \\
\mathcal{M}, w \models \neg\phi & \text{iff} & \mathcal{M}, w \not\models \phi \\
\mathcal{M}, w \models \phi \wedge \psi & \text{iff} & \mathcal{M}, w \models \phi \text{ and } \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models K_i\phi & \text{iff} & \forall w' \in W : w \sim_i w' \Rightarrow \mathcal{M}, w' \models \phi \\
\mathcal{M}, w \models [\psi]\phi & \text{iff} & \mathcal{M}, w \models \psi \Rightarrow \mathcal{M}|_\psi, w \models \phi
\end{array}
$$

where $\mathcal{M}|_\phi = (W|_\phi, (\sim_i |_\phi)_{i \in N}, V|_\phi)$ is defined as:

$$
\begin{array}{lll}
W|_\phi & = & \{v \mid \mathcal{M}, v \models \phi\} \\
\sim_i |_\phi & = & \sim_i \cap \, W|_\phi \times W|_\phi \\
V|_\phi(p) & = & V(p) \cap W|_\phi.
\end{array}
$$

Definition 2.3 (Axiomatization)**.** The axiomatization of PAL extends that of multi-agent epistemic logic with the necessitation rule for PAL-operators $[A]$, and the following reduction axioms:

$$
\begin{array}{lll}
\langle A \rangle p & \leftrightarrow & A \wedge p \quad (\text{with } p \in P) \\
\langle A \rangle \neg\phi & \leftrightarrow & A \wedge \neg\langle A \rangle \phi \\
\langle A \rangle (\phi \vee \psi) & \leftrightarrow & \langle A \rangle \phi \vee \langle A \rangle \psi \\
\langle A \rangle K_i\phi & \leftrightarrow & A \wedge K_i(A \rightarrow \langle A \rangle \phi)
\end{array}
$$

It is easy to check the soundness of these axioms and the rule. Also, notice that, with the reduction axioms, every formula of $\mathcal{L}_{PAL}$ reduce to an equivalent formula of $\mathcal{L}_{el}$. Thus, given the completeness of epistemic logic, we have the following:

Theorem 2.4 (In e.g. [4])**.** The axiomatization of PAL is complete.

2.2 $TPAL$

Now as we mentioned in the introduction, PAL presupposes that whatever is true is announcable. This is already evident in the truth definition for the PAL-operator. The statement "The public announcement A can be made after which ϕ" is true iff A is true and ϕ is true in the model without non-A-worlds. Thus, the condition for the announcability is only the truth of what is to be announced. We can also see this assumption at work in the right hand side of the reduction axioms (except for disjunction). There, we see the occurrences of A's as one of the conjuncts.

To overcome this unrealistic feature of the system, we model constraints on the set of "permissible" sequences of announcements by incorporating into the DEL-setting of PAL the framework of epistemic temporal logic (ETL), which is another trend of modelling agents' epistemic states over informational updates that has been developed (see e.g. [9, 15]). Here we present the semantic setting in [6].

Definition 2.5 (Language)**.** The language $\mathcal{L}_{TPAL}$ of $TPAL$ extends $\mathcal{L}_{el}$ with the operator $\langle A \rangle$ where $A \in \mathcal{L}_{el}$. Let P be the set of propositional letters. The formulas are defined by:

$$\phi \ ::= \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid K_i\phi \mid \langle A \rangle \phi$$

where $p \in P$ and $A \in \mathcal{L}_{el}$. The dual operator $[A]$ is defined by $\neg\langle A \rangle \neg$. The other booleans and duals are defined in the usual way.

For the semantic setting of $TPAL$, we first give some preliminary definitions. Given a set Σ^*, we define Σ^* as the set of (possibly empty) finite sequences of the elements in Σ. We denote the empty sequence by λ. For a sequence $\sigma = \phi_1\phi_2...\phi_n \in \Sigma^*$ (when $n = 0$, σ is empty), we define the length $len(\sigma)$ of σ to be n, and σ_m by $\sigma_m = \lambda$ (empty sequence) if $m = 0$; $\sigma_m = \phi_1...\phi_m$ otherwise. For a pair of sequences σ, τ, we put $\sigma \preceq \tau$ if σ is an initial segment of τ. Also we write $\sigma\tau$ for the concatenation of the two sequences. For an $S \subseteq \Sigma^*$, we define $FinPre_{\{-\lambda\}}(S) = \{h \mid \exists h' \in S : h \preceq h' \text{ and } h \neq \lambda\}$.

Let $Ptcl((\mathcal{L}_{el})^*) = \{\mathcal{E} | \mathcal{E} = FinPre_{\{-\lambda\}}(A) \text{ for some } A \subseteq (\mathcal{L}_{el})^*\}$. Given an epistemic model $\mathcal{M}$, *a PAL-protocol* on $\mathcal{M}$ is a function that maps every point in $\mathcal{M}$ to some element in $Ptcl((\mathcal{L}_{el})^*)$. Given a point w, $f(w)$ represents the sequences of permissble announcements at w.

Definition 2.6. Let $\mathcal{M} = (W, \sim_i, V)$ be an epistemic model and f, a PAL-protocol on $\mathcal{M}$. Given a sequence $\sigma = \phi_1...\phi_n$ $(n \geq 1)$, a model $\mathcal{M}^{\sigma,f} = (W^{\sigma,f}, \sim^{\sigma,f}, V^{\sigma,f})$ is defined by:

- $W^{\sigma_0,f} := W$.
- $W^{\sigma_{m+1},f} := \{w\sigma_{m+1} | w \in W \wedge \mathcal{M}^{\sigma_m,f}, w\sigma_m \models \phi_{m+1} \wedge \sigma_{m+1} \in f(w)\}$
- For each $w\sigma_{m+1}, v\sigma_{m+1} \in W^{\sigma_{m+1},f}$, $w\sigma_{m+1} \sim_i^{\sigma_{m+1},f} v\sigma_{m+1} \Leftrightarrow w \sim_i v$.
- For each prop. letter p, $V^{\sigma_{m+1},f}(p) = \{w\sigma_{m+1} \in W^{\sigma_{m+1},f} | w \in V(p)\}$.

Now for the semantic setting of $TPAL$-formulas, we generate ETL-models from epistemic models and PAL-protocols in the following way.

Definition 2.7 (DEL-Generated ETL-Models)**.** Given an epistemic model $\mathcal{M}$ and a PAL-protocol f on $\mathcal{M}$, the ETL-model $Forest(\mathcal{M}, f) = (H, \approx_i, U)$ is generated as follows:

- H: $\{h | h \in W^{\sigma,f} \text{ for some } \sigma \in \bigcup_{w \in W} f(w)\}$.
- $\approx_i$: for all $h, h' \in H$, $h \approx_i h' \Leftrightarrow h \sim_i^{\sigma,f} h'$, where $h = w\sigma$ and $h' = v\sigma$ for some $\sigma \in \mathcal{L}^*_{PAL}$.
- U: for every prop. letter p, $h \in U(p) \Leftrightarrow h \in V^{\sigma,f}(p)$, where $h = w\sigma$ for some $\sigma \in \mathcal{L}^*_{PAL}$.

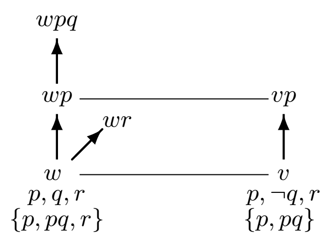
We call *histories* the elements in H. We define $Forest(PAL)$ as the set of all ETL-models generated from epistemic models and PAL-protocols.

Definition 2.8 (Truth)**.** Given an ETL-model $\mathcal{H} = (H, \sim_i, V)$ generated from some epistemic model and PAL-protocol, and a history $h \in H$, the truth at h of a $TPAL$-formula is inductively defined as follows:

$$
\begin{array}{lll}
\mathcal{H}, h \models p & \text{iff} & h \in V(p) \quad (\text{with } p \in P) \\
\mathcal{H}, h \models \neg\phi & \text{iff} & \mathcal{H}, h \not\models \phi \\
\mathcal{H}, h \models \phi \wedge \psi & \text{iff} & \mathcal{H}, h \models \phi \text{ and } \mathcal{H}, h \models \psi \\
\mathcal{H}, h \models K_i\phi & \text{iff} & \forall h' \in H : h \sim_i h' \Rightarrow \mathcal{H}, h' \models \phi \\
\mathcal{H}, h \models \langle\psi\rangle\phi & \text{iff} & h\psi \in H \text{ and } \mathcal{H}, h\psi \models \phi
\end{array}
$$

Consistency, satisfiability, validity etc. are defined in a familiar way.

Example 2.9. To see how the semantics of $TPAL$ works, let us see one simple example. Let $\mathcal{M}$ consists of two worlds, w, v, indistinguishable and the valuation that makes p, r true at both worlds but that makes q true only at w. Also let a PAL-protocol f be such that $f(w) = \{p, pq, r\}$ and $f(v) = \{p, pq\}$. The generated ETL-model $Forest(\mathcal{M}, f)$ will be visualized as follows:



where the horizontal lines represent the indistinguishability relation and the arrows represent the available announcements at each histories. In this model $\mathcal{H} = Forest(\mathcal{M}, f)$, we, for instance, have $\mathcal{H}, w \models \langle r\rangle Kp$, since we have $\mathcal{H}, wr \models p$ and thus $\mathcal{H}, wr \models Kp$.

Example 2.10. We can encode into a PAL-protocol the conversational constraints in the above MC case in the following way. In MC, the father first utters the initial announcement and then the children answers at every round whether they know they are dirty. There must not be other conversational events in MC. Define $(\phi)^+$ and ϕ^- respectively as ϕ and $\neg\phi$. Let S be the set of triples where each element is either $+$ or $-$. Given an $s = (s_1, s_2, s_3) \in S$, put $A_s := \bigwedge_{i=0}^{3}(K_i d_i)^{s_i}$. Then define a PAL-protocol f_{MC} as

$$f(w) = FinPre_{\{-\lambda\}}(\{(\bigvee_{i=1}^{3} d_i)A_{\sigma_1}...A_{\sigma_n} | \sigma_i \in S \text{ for all } i\})$$

for every w in the epistemic model. $\bigvee_{i=1}^{3} d_i$ represents the father's initial announcement and the following statements A_{σ_i}, the children's answers.

Next we see some semantic features of $TPAL$ to refer to later when we introduce the generalized PAL-operators. First, $TPAL$-formulas only describe the "future" states in the tree models up to its *depth*.

Definition 2.11 (t-depth)**.** The t-depth $d(\phi)$ of a $TAPAL$ formula ϕ is defined as follows:

- $d(p) = 0$ with $p \in P$
- $d(\neg\phi) = d(\phi)$
- $d(\phi \wedge \psi) = max(d(\phi), d(\psi))$
- $d(K_i\phi) = d(\phi)$
- $d(\langle A\rangle\phi) = 1 + d(\phi)$

Given a protocol f on $\mathcal{M}$ and a sequence $\sigma \in (\mathcal{L}_{el})^*$, we define a protocol $f_k^{\sigma<}$ on $\mathcal{M}^{\sigma,f}$ so that $f_k^{\sigma<}(w\sigma) = \{\tau | \sigma\tau \in f(w) \text{ and } len(\tau) \leq k\}$ for all $w\sigma$ in $\mathcal{M}^{\sigma,f}$. This represents which sequences of formulas with length k or less are announcable after σ. Also, we define $f^{\sigma<}(w\sigma) = \{\tau | \sigma\tau \in f(w)\}$ when not stating the upper bound on the length.

Propositoin 2.12 (In [6].)**.** For all w in $\mathcal{M}$ and $\sigma \in (\mathcal{L}_{el})^*$,

$$Forest(\mathcal{M}, f), w\sigma \models \phi \text{ iff } Forest(\mathcal{M}^{\sigma,f}, f_{d(\phi)}^{\sigma<}), w\sigma \models \phi.$$

Also,

$$Forest(\mathcal{M}, f), w\sigma \models \phi \text{ iff } Forest(\mathcal{M}^{\sigma,f}, f^{\sigma<}), w\sigma \models \phi.$$

Second, we see that the truth of $TPAL$-formulas depends only on the sequences of announcements that are *relevant* to them. Let $sub^a(\phi)$ be the set of subformulas of ϕ that are in $\mathcal{L}_{el}$. Given a value $f(w)$, define $(f(w))_{sub^a(\phi)}$ by the set

$$\{\sigma \in f(w) | \text{For every element } \theta \text{ in } \sigma, \theta \in sub^a(\phi)\}.$$

This set represents permissible sequences of announcements at w that only consist of subformulas of ϕ.

Propositoin 2.13 (In [6])**.** Let $(f(v))_{sub^a(\phi)} = (g(v))_{sub^a(\phi)}$ for all v in $\mathcal{M}$. For all w in $\mathcal{M}$,

$$Forest(\mathcal{M}, f), w \models \phi \text{ iff } Forest(\mathcal{M}, g), w \models \phi.$$

Now we present the axiomatization of $TPAL$.

Definition 2.14 (Axiomatization)**.** The axiomatization of $TPAL$ adds to multi-agent epistemic logic the following axioms and the necessitation rule for $[A]$:

R1 $\langle A\rangle p \leftrightarrow \langle A\rangle\top \wedge p$

R2 $\langle A\rangle\neg\phi \leftrightarrow \langle A\rangle\top \wedge \neg\langle A\rangle\phi$

R3 $\langle A\rangle(\phi \wedge \psi) \leftrightarrow \langle A\rangle\phi \wedge \langle A\rangle\psi$

R4 $\langle A\rangle K_i\phi \leftrightarrow \langle A\rangle\top \wedge K_i(\langle A\rangle\top \rightarrow \langle A\rangle\phi)$

A1 $[A](\phi \to \psi) \to ([A]\phi \to [A]\psi)$

A2 $\langle A\rangle\top \to A$

Readers are invited to check the soundness of these axioms and the rule.

Note that, with the semantics of $TPAL$, $\langle A\rangle\top$ means that A is announcable. More precisely, it means that concatenating A to the current history h results in the set of permissible sequences of announcements. Thus, the axioms represents that the truth of A does not imply that A is announcable in $TPAL$. Because of this feature, the reduction axioms R1-4 does not reduce the formulas of $TPAL$ to the equivalent formulas of epistemic logic unlike in the case of PAL.

Since the standard completeness proof via reduction (for the proof via reduction in PAL, see e.g. in [one lonely]) does not work in $TPAL$, we need to give a proof independently for the completeness $TPAL$. This can be done as is in [6]. (We will see similar construction in the next section.)

Theorem 2.15 (In [6])**.** $TPAL$ is complete with the class $Forest(PAL)$.

3 Generalized PAL-Operators $\Diamond$ and $\Diamond^*$

3.1 Distinguishing two generalized operators

Now we would like to consider the generalized PAL-operator that quantifies over public announcements. This is done in the originial PAL-setting by introducing the operator $\Diamond$, where $\Diamond\phi$ reads as "There is some announcement after which ϕ." Given an epistemic model $\mathcal{M}$, the semantics of this operator is given by:

$$\mathcal{M}, w \models \Diamond\phi \text{ iff there is some } \psi \in \mathcal{L}_{el} \text{ such that } \mathcal{M}, w \models \langle\psi\rangle\phi.$$

The extension of PAL with this operator is called $APAL$ and studied in [1].

Now, to consider such a generalized operator in the $TPAL$-setting, we have to be cautious about the following fact. In PAL, sequences of announcements are identified with some single announcements by

$$\langle\phi\rangle\langle\psi\rangle\theta \leftrightarrow \langle\langle\phi\rangle\psi\rangle\theta.$$

However, in $TPAL$, this is not the case. $TPAL$ invalidates the schema, since the corresponding single announcements may not be available even if sequences of announcements are available. Thus, we have to distinguish single announcements and sequences of announcements in the $TPAL$-setting.

This consideration motivates us to introduce the following two generalized operators. Let $\mathcal{H} = Forest(\mathcal{M}, f)$ be an ETL-model generated from an epistemic model $\mathcal{M}$ and a PAL-protocol f. First, we define the generalized operator for single announcements:

$$\mathcal{H}, h \models \Diamond\phi \Leftrightarrow \text{ There is a formula } \psi \text{ such that } h\psi \text{ in } \mathcal{H} \text{ and } \mathcal{H}, h\psi \models \phi.$$

Thus $\Diamond\phi$ reads as "Some single public announcement can be made after which ϕ holds". We denote the dual of $\Diamond$ by $\Box$, where $\Box\phi$ reads as "after every single public announcement, ϕ." Next, we define the generalized operator for sequences of announcements:

$$\mathcal{H}, h \models \Diamond^* \phi \Leftrightarrow \text{ There is a sequence } \sigma \text{ such that } h\sigma \text{ in } \mathcal{H} \text{ and } \mathcal{H}, h\sigma \models \phi.$$

We allow σ to be possibly empty. $\Diamond^* \phi$ reads as "Some sequence of announcements can be made after which ϕ holds." We denote the dual of $\Diamond^*$ by $\Box^*$, where $\Box^* \phi$ reads as "after every sequences of public announcements, ϕ". We call $TAPAL$ the system that extends $TPAL$ with $\Diamond$, and $TASPAL$, the system with both $\Diamond$ and $\Diamond^*$.

Definition 3.1 (Languages)**.** The language $\mathcal{L}_{TASPAL}$ extends $\mathcal{L}_{TPAL}$ with the operator $\Diamond$. The formulas are defined by:

$$\phi \ ::= p \mid \top \mid \neg\phi \mid \phi \wedge \phi \mid K_i\phi \mid \langle A \rangle \phi \mid \Diamond \phi \mid \Diamond^* \phi$$

where $p \in P$ and $A \in \mathcal{L}_{el}$. The dual operators $\Box$ and $\Box^*$ are defined by $\neg\Diamond\neg$ and $\neg\Diamond^*\neg$ respectively. The other booleans and duals are defined in the usual way.

The language $\mathcal{L}_{TAPAL}$ is the fragment of $\mathcal{L}_{TASPAL}$ without the operator $\Diamond^*$.

Example 3.2. It is a straighforward task to verify the formulas, $\langle \bigvee_{i=0}^{3} p_i \rangle \Diamond K_i d_i$ is now false in the MC case mentioned in Example 2.10. It is also straightforward to verify that other true statements about the MC case also correspond to the solution of MC, e.g. $\langle \bigvee_{i=0}^{3} p_i \rangle \Diamond\Diamond K_1 d_1$ (1 knows that he is dirty after two question rounds), $\langle \bigvee_{i=0}^{3} p_i \rangle \neg \Diamond^* (\bigvee_{i \neq j} (\neg K_i d_i \wedge K_j d_j)$ (Each child comes to know that he is dirty at the same time.), etc.

3.2 Semantic Results

Now we will see how these operators work semantically. First, we will observe that the operators, $\Diamond$ in $APAL$, $\Diamond$ in $TAPAL$, and $\Diamond^*$, behave in different ways. Consider the following semantic properties:

1. $\models \Box\phi \rightarrow \phi$
2. $\models \Box\phi \rightarrow \Box\Box\phi$
3. $\models \Box\Diamond\phi \rightarrow \Diamond\Box\phi$
4. $\models \Diamond\Box\phi \rightarrow \Box\Diamond\phi$

Propositoin 3.3. (Semantic Difference)

(A) All of the properties 1-4 hold in $APAL$.

(B) None of the properties 1-4 holds in $TAPAL$.

(C) The properties 1-2 hold, but 3 and 4 don't, in $TASPAL$. (when $\Diamond$ and $\Box$ are replaced with $\Diamond^*$ and $\Box^*$ respectively)

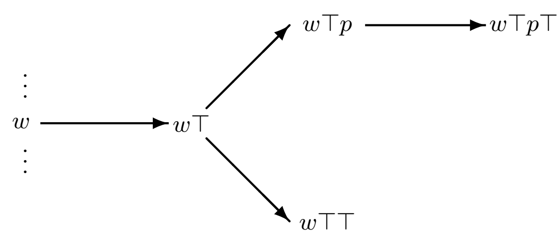
Proof. (A) The proofs of the properties in $APAL$ are in [1].

(B) We give the counterexamples as follows:

1. Let w be in $\mathcal{M}$ and $f(w) = \emptyset$. Then, clearly, $Forest(\mathcal{M}, f), w \models \Box \bot$, but $Forest(\mathcal{M}, f), w \not\models \bot$.

2. Let w be in $\mathcal{M}$ and $f(w) = \{\top, \top\top\}$. Put $\mathcal{H} = Forest(\mathcal{M}, f)$. Then, we have $\mathcal{H}, w\top \models \langle\top\rangle\top$, but $\mathcal{H}, w\top\top \not\models \langle\top\rangle\top$. Therefore, we have $\mathcal{H}, w \models \Box\langle\top\rangle\top$ but $\mathcal{H}, w \not\models \Box\Box\langle\top\rangle\top$.

3. Let $\mathcal{M}, w \models p$. Define $f(w) = \{\top, \top\top, \top p, \top p\top\}$. The model $\mathcal{H} = Forest(\mathcal{M}, f)$ can be represented by the figure below. Here we have $\mathcal{H}, w\top p \models \langle\top\rangle\top$, but $\mathcal{H}, w\top\top \not\models \langle\top\rangle\top$. Therefore, we have $\mathcal{H}, w \models \Box\Diamond\langle\top\rangle\top$, but $\mathcal{H}, w \models \Box\Diamond\neg\langle\top\rangle\top$, i.e. $\mathcal{H}, w \not\models \Diamond\Box\langle\top\rangle\top$.



4. In the above model, $\mathcal{H}, w\top p \models \Box\top$, which yields $\mathcal{H}, w\top \models \Diamond\Box\top$, but $\mathcal{H}, w\top\top \not\models \Diamond\top$, which yields $\mathcal{H}, w\top \not\models \Box\Diamond\top$.

(C) We prove 1 and 2, and give counterexamples for 3 and 4.

1. Assume that $\mathcal{H}, h \models \Box^*\phi$. By the semantics, $\mathcal{H}, h\sigma \models \phi$ for all $h\sigma$ in $\mathcal{H}$. Take the empty sequence for σ. Then, $\mathcal{H}, h \models \phi$.

2. We prove the contraposition. Assume that $\mathcal{H}, h \models \Diamond^*\Diamond^*\phi$. Then there is some sequences σ, τ such that $\mathcal{H}, h\sigma\tau \models \phi$. Since $\sigma\tau$ is also a sequence, it follows that $\mathcal{H}, h \models \Diamond^*\phi$.

3. Let $\mathcal{M}, w \models p$. Define $f(w) = \{\top, \top p, \top p\top, \top p\top p, ...\}$. Let $\mathcal{H}$ be $Forest(\mathcal{M}, f)$. We claim that, for every h in $\mathcal{H}$, there exists $\sigma, \sigma' \in f(w)$ such that $\mathcal{H}, h\sigma \models \langle\top\rangle\top$ and $\mathcal{H}, h\sigma' \not\models \langle\top\rangle\top$. To see this, note that every h ends with either $\top$ or p. If h ends with $\top$, then put $\sigma = p$ and $\sigma' = \emptyset$; if h ends with p, then put $\sigma = \emptyset$ and $\sigma' = \top$. This fact implies $\mathcal{H}, w \models \Box\Diamond\langle\top\rangle\top$ and $\mathcal{H}, w \models \Box\Diamond\neg\langle\top\rangle\top$. Thus, this model is a counterexample against 3.

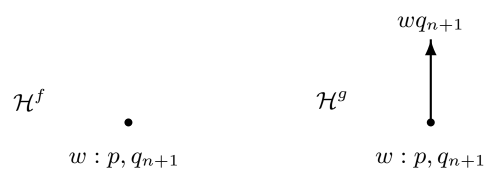
4. The model for B4 similarly works.

$\Box$

Now we will see the results concerning expressivity of $TAPAL$ and $TASPAL$. First, since the operators $\Diamond$ and $\Diamond^*$ implicitly refers to (sequences of) announcements in ETL-models by quantification, we do not have results similar to Proposition 2.13 for $TAPAL$ or $TASPAL$. This feature of the operators add expressivity to the system as in the following result.

Propositoin 3.4. Both $TAPAL$ and $TASPAL$ are strictly more expressive than $TPAL$. That is, $TPAL \subset TAPAL$ and $TPAL \subset TASPAL$.

Proof. Since $TAPAL \subseteq TASPAL$ by definition, it suffices to show that $TPAL \subset TAPAL$. For this, consider the formula $\Diamond p$. Assume toward contradiction that this formula is equivalent to some $TPAL$-formula ψ. Since $TPAL$-formulas are finite, there are only a finite number of propositional letters, $q_1, q_2, ..., q_n$, used in ψ. Let q_{n+1} be a propositional letter that is distinct from all $q_1, q_2, ..., q_n$, and $\mathcal{M}$, an epistemic model with only a state w at which p, q_{n+1} are both true. Then define f, g be $f(w) = \emptyset$ and $g(w) = \{q_{n+1}\}$. Now consider $\mathcal{H}^f = Forest(\mathcal{M}, f)$ and $\mathcal{H}^g = Forest(\mathcal{M}, g)$.



Since $(f(w))_{sub^a(\psi)} = (g(w))_{sub^a(\psi)}$, it follows from Proposition 2.13 that ψ has the same value at $\mathcal{H}^f, w$ and $\mathcal{H}^g, w$. However, clearly $\mathcal{H}^f, w \not\models \Diamond p$ and $\mathcal{H}^g, w \models \Diamond p$. This is a contradiction. Thus $TPAL \subset TAPAL$. □

On the other hand, since $\Diamond$ and $\Diamond^*$ are future-looking in the sense that the truth value of the formulas does not depend on the nodes below a point of evaluation, we can obtain a result similar to Proposition 2.12 also for $TAPAL$.

Define the t-depth $d(\phi)$ of $TAPAL$-formulas in the same as in Definition 2.11 except that we now add

$$d(\Diamond\phi) = 1 + d(\phi)$$

as the extra item.

Propositoin 3.5. Let w in $\mathcal{M}$, $\sigma \in (\mathcal{L}_{el})^*$, and $\phi \in \mathcal{L}_{TAPAL}$. Then,

$$Forest(\mathcal{M}, f), w\sigma \models \phi \text{ iff } Forest(\mathcal{M}^{\sigma,f}, f^{\sigma<}_{d(\phi)}), w\sigma \models \phi.$$

Also,

$$Forest(\mathcal{M}, f), w\sigma \models \phi \text{ iff } Forest(\mathcal{M}^{\sigma,f}, f^{\sigma<}), w\sigma \models \phi.$$

Proof. By induction on ϕ. □

For $TASPAL$-formulas, we cannot place the explicit upper bound by the t-depth. This is because the operator $\Diamond^*$ quantifies over all finite sequences. Thus, we only have:

Propositoin 3.6. Let w in $\mathcal{M}$, $\sigma \in (\mathcal{L}_{el})^*$, and $\phi \in \mathcal{L}_{TASPAL}$. Then,

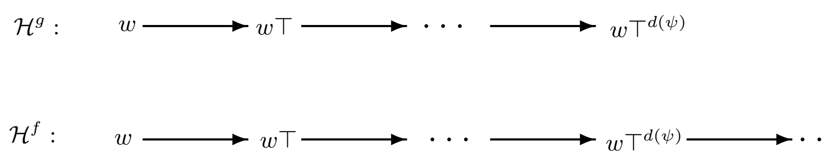
$$Forest(\mathcal{M}, f), w\sigma \models \phi \text{ iff } Forest(\mathcal{M}^{\sigma,f}, f^{\sigma<}), w\sigma \models \phi.$$

Proof. By induction on ϕ. □

The difference of $TPAL$ and $TASPAL$ in the above two propositions feature contributes to the following expressivity result.

Propositoin 3.7. $TASPAL$ is strictly more expressive than $TAPAL$. That is, $TAPAL \subset TASPAL$.

Proof. Consider $\Box^*\langle\top\rangle\top$. Assume toward contradiction that this formula is equivalent to some $TAPAL$-formula ψ. Let $\mathcal{M}$ be an epistemic model with only a state w. Let us denote as $\top^k$ the sequence of k $\top$'s. Define f, g be such that$f(w) = \{\top^i | 0 \leq i \leq d(\psi)\}$ and $g(w) = \{\top^i | i \in \mathbf{N}\}$. Now consider $\mathcal{H}^f = Forest(\mathcal{M}, f)$ and $\mathcal{H}^g = Forest(\mathcal{M}, g)$.



Since $\mathcal{H}^f = Forest(\mathcal{M}, g^{\lambda<}_{d(\psi)})$ with λ the empty sequence, it follows from Proposition 3.5 that ψ has the same value at $\mathcal{H}^f, w$ and $\mathcal{H}^g, w$. On the other hand, $\mathcal{H}^f, w\top^{d(\psi)} \not\models \langle\top\rangle\top$, which implies $\mathcal{H}^f, w \not\models \Box^*\langle\top\rangle\top$, whereas clearly $\mathcal{H}^g, w \models \langle\top\rangle\top$. □

Corollary 3.8. $TPAL \subset TAPAL \subset TASPAL$.

The expressive power of $\Diamond$ and $\Diamond^*$ also render the systems noncompact.

Propositoin 3.9. Neither $TAPAL$ nor $TASPAL$ are compact.

Proof. Consider the set $\Gamma = \{\neg\langle\theta\rangle p | \theta \in \mathcal{L}_{el}\} \cup \{\Diamond p\}$. Γ is clearly unsatisfiable by the semantic definition of $\Diamond$. For p must be true in the epistemic model that generates an ETL-model to make $\Diamond p$ true, but, then, to make $\{\neg\langle\theta\rangle p | \theta \in \mathcal{L}_{TAPAL}\}$ satisfied, we have to make the PAL-protocol empty. However, any finite subset Γ' of Γ is satisfiable. If $\Diamond p \not\in \Gamma'$, let the protocol be empty. If $\Diamond p \in \Gamma'$, then take a propositional letter q that does not occur in Γ' and define a model so that, at a given point, p and q are true and the protocol allows the announcement q. □

Note that the non-compactness still plagues even when we restrict language to $\Diamond^*$ (without $\Diamond$). This is easy to verify when we consider the set $\bigcup_{i=0}^{\infty} \Gamma_i \cup \{\Diamond^* p\}$, where $\Gamma_i = \{\neg\langle\theta_0\rangle ... \langle\theta_i\rangle p | \theta_j \in \mathcal{L}_{el}\ (0 \leq j \leq i)\}$.

4 Axiomatizations: $TAPAL$ and $TASPAL$

Now we present the complete axiomatizations for $TAPAL$ (and $TASPAL$ in the next subsection). We denote by P_ϕ the set of propositional letters used in a formula ϕ. We also denote the set of propositional letters used in sequences of formulas, sets of formulas, in the same way by putting the sequences, sets etc. as the subscripts. Also, given a sequence $\sigma = \phi_1 ... \phi_n$ of formulas, we denote by $\langle\sigma\rangle$ the sequence $\langle\phi_1\rangle ... \langle\phi_n\rangle$. Particularly, when $n = 0$, $\langle\sigma\rangle$ is just empty. We use a similar notation for the dual operators: $[\sigma]$ is $[\phi_1]...[\phi_n]$.

4.1 TAPAL

Definition 4.1. The axiomatization of $TAPAL$ adds the following axiom Gen and the inference rule $R(\Box)$ to that of $TPAL$.

Gen $\langle\chi\rangle\phi \to \Diamond\phi$ for any $\chi \in \mathcal{L}_{el}$

$R(\Box)$ If $\vdash \phi \to [\sigma][p]\psi$ with $\sigma \in (\mathcal{L}_{el})^*$, then $\vdash \phi \wedge [\sigma]\Box\phi$.

where $p \notin P_\phi \cup P_\psi \cup P_\sigma$.

The role of Gen is clear by the truth definition. On the other hand, to see what $R(\Box)$ does, an analogy may be helpful between $R(\Box)$ and the first-order rule:

$$\text{If } \vdash \phi \to \psi \text{ with no occurrence of } x \text{ in } \phi, \text{ then infer } \vdash \phi \to \forall x\psi.$$

In fact, as we will see below in the completeness proof, the use of $R(\Box)$ is very similar to the use of this first-order rule in the completeness proof. See below.

For the proof of the soundness of $R(\Box)$, we need the following. First, we denote as $\phi[\alpha_0 \mapsto \beta_0, \alpha_1 \mapsto \beta_1, ...]$ the result of the uniform substitutions $\alpha_i \mapsto \beta_i$ of β_i for α_i in a formula ϕ $(0 \leq i)$. We use this notation applied also to sets and sequences of formulas to mean the result of performing the substitutions in all the formulas in the sets and sequences.

Let $Q = \{p_n | n \in \mathbb{N}\}$ be a countable subset of the set of propositional letters, and let θ be a sentence in the epistemic language $\mathcal{L}_{ep}$ with $P_\theta \cap Q = \emptyset$. Take a $TAPAL$-model $\mathcal{H} = Forest(\mathcal{M}, f)$ generated by an epistemic model $\mathcal{M} = (F, V)$ and a PAL-protocol f. Fix a hitory $h_0 \in \mathcal{H}$ and put $h_0 = w\sigma_0$. Then construct a $TAPAL$-model $\mathcal{H}' = Forest((F, V'), f')$ in the following way:

- $V'(p_0) = \{v \in W | \mathcal{H}, v \models \langle\sigma_0\rangle\theta\}$.
- $V'(p_{i+1}) = V(p_i)$ for all $i \in \mathbf{N}$.
- $f'(w) = \{\tau' | \tau \in f(w)\}$ where we define τ' for every $\tau \in f(w)$ by
  - $\tau' = \sigma_0[(p_i \mapsto p_{i+1})_{i \in \mathbf{N}}]p_0\upsilon[(p_i \mapsto p_{i+1})_{i \in \mathbf{N}}]$ if $\tau = \sigma_0\theta\upsilon$.
  - $\tau' = \tau[(p_i \mapsto p_{i+1})_{i \in \mathbf{N}}]$ otherwise.

Lemma 4.2. For every formula $\phi \in \mathcal{L}_{TAPAL}$ and $h \in \mathcal{H}$, $\mathcal{H}' = Forest((F, V'), PAL')$ satisfies the following:

1. $\mathcal{H}, h \models \phi[p_0 \mapsto \theta, (p_{i+1} \mapsto p_i)_{i \in \mathbf{N}}] \Leftrightarrow \mathcal{H}', h' \models \phi$.
2. $\mathcal{H}, h \models \phi \Leftrightarrow \mathcal{H}', h' \models \phi[(p_i \mapsto p_{i+1})_{i \in \mathbf{N}}]$.

where $h' = w\tau'$ for all $h = w\tau \in \mathcal{H}$.

Proof. By induction on ϕ. □

Propositoin 4.3 (Soundness of $R(\Box)$)**.** If $\psi \wedge \langle\sigma\rangle\Diamond\phi$ is satisfiable where σ is a sequence of formulas and $p \notin P_\phi \cup P_\psi \cup P_\sigma$, then $\psi \wedge \langle\sigma\rangle\langle p\rangle\phi$ is satisfiable.

Proof. Assume that $Forest((F, V), PAL), h \models \psi \wedge \langle\sigma\rangle\Diamond\phi$. Set $Q := P\backslash(P_\phi \cup P_\psi \cup P_\sigma)$ and $\theta := \top$ in the above lemma. Then, we obtain $Forest((F, V'), f'), h' \models \psi \wedge \langle\sigma\rangle\Diamond\phi$ with V', f', h' as constructed in the proof of the lemma. This implies by truth definitions that $Forest((F, V'), f'), h' \models \psi \wedge \langle\sigma\rangle\langle\eta\rangle\phi$ for some formula η. Here we can replace the occurrences in η of p with $\top$, since the valuation $V'(p) = V(\theta) = \top$. Thus, we assume that η does not contain the occurrences of p. Then set $Q := P\backslash(P_\phi \cup P_\psi \cup P_\sigma \cup P_\eta)$, $p_0 = p$ and $\theta := \eta$ again in the above lemma to obtain the model such that $Forest((F, V''), f''), h'' \models \psi \wedge \langle\sigma\rangle\langle p\rangle\phi$. □

Now we prove the (weak) completeness. For this, we construct the canonical model by following the construction in [6] for the completeness of $TPAL$. The difference, though, for $TAPAL$ is that the construction is not from the set of all the maximal constent set, but from the set of the maximal consistent sets Γ with the following property: for every sentence of the form $\langle\sigma\rangle\Diamond\phi$ with σ a sequence of formulas, if $\langle\sigma\rangle\Diamond\phi \in \Gamma$, then there is some formula θ such that $\langle\sigma\rangle\langle\theta\rangle\psi \in \Gamma$.

The reason for the construction is to make sure that there is a formula that witnesses $\Diamond$ in every formula in a given maximally consistent set. Here the above analogy between $R(\Box)$ and the first-order rule comes back again. In the proof below, when we construct a maximal consistent set from a consistent formula, we add witnessing formulas for the formulas of the above form. The consistency of the resulting set with witnessing formulas will be guaranteed by the rule $R(\Box)$, and this is very similar to the way that the first-order rule in question (or its equivalent) is used in the completeness proof of first-order logic.

Definition 4.4 (saturation wrt $\Diamond$)**.**) A set Σ of formula is *saturated with respect to* $\Diamond$, if, for every sentence of the form $\langle\sigma\rangle\Diamond\phi$ with σ a sequence of formulas, $\langle\sigma\rangle\Diamond\phi \in \Sigma$ implies that there is some formula θ such that $\langle\sigma\rangle\langle\theta\rangle\phi \in \Sigma$.

Lemma 4.5. (Lindenbaum) Every consistent $TAPAL$-formula ϕ can be expanded to a maximal consistent set saturated with respect to $\Diamond$.

Proof. Let $\alpha_1, .\alpha_2...$ be an enumeration of the TAPAL-formulas such that $\alpha_1 = \phi$. We construct a sequence $\Sigma_0, \Sigma_1, ...$ of sets as follows:

- $\Sigma_0 = \emptyset$

- If $\Sigma_n \cup \{\alpha_n\}$ is inconsistent, $\Sigma_{n+1} = \Sigma_n$.
- If $\Sigma_n \cup \{\alpha_n\}$ is consistent and α_n is not of the form $\langle\sigma\rangle\Diamond\psi$, $\Sigma_{n+1} = \Sigma_n \cup \{\alpha_n\}$.
- If $\Sigma_n \cup \{\alpha_n\}$ is consistent and α_n is of the form$\langle\sigma\rangle\Diamond\psi$, $\Sigma_{n+1} = \Sigma_n \cup \{\langle\sigma\rangle\Diamond\psi, \langle\sigma\rangle\langle p\rangle\psi\}$ for a propositional letter $p \notin P_\psi \cup P_\sigma \cup P_{\Sigma_n}$. Such a propositional letter exists since Σ_n is finite and we have a countable set of propositional letters.

We show by induction that Σ_n is consistent for $n \geq 1$. The base case is given by the assumption that ϕ is consistent. Assume that Σ_n is consistent for an arbitrary n. If $\Sigma_n \cup \{\alpha_n\}$ is inconsistent, Σ_{n+1} is consistent since $\Sigma_{n+1} = \Sigma_n$. Thus, assume that $\Sigma_n \cup \{\alpha_n\}$ is consistent. By the construction, Σ_{n+1} is clearly consistent if α_n is not of the form $\langle\sigma\rangle\Diamond\psi$. We claim that Σ_{n+1} is also consistent, if α_n is of the form. For suppose otherwise. Then, there must be some formulas $\psi_1, \psi_2, ..., \psi_l \in \Sigma_m \cup \{\langle\sigma\rangle\Diamond\psi\}$ such that

$$\vdash (\psi_1 \wedge ... \wedge \psi_l) \rightarrow \neg\langle\sigma\rangle\langle p\rangle\psi.$$

However, this implies

$$\vdash (\psi_1 \wedge ... \wedge \psi_l) \rightarrow [\sigma][p]\neg\psi.$$

Since p is chosen so that $p \notin P_\psi \cup P_\sigma \cup P_{\Sigma_n}$, we can apply $R(\Box)$ to obtain

$$\vdash (\psi_1 \wedge ... \wedge \psi_l) \rightarrow [\sigma]\Box\neg\psi$$

This gives us $\Sigma_m \vdash [\sigma]\Box\neg\psi$ and $\Sigma_m \vdash \neg\langle\sigma\rangle\Diamond\psi$. However this contradicts the assumption that $\Sigma_n \cup \{\alpha_n\}$ is consistent.

Now take $\Sigma' = \bigcup_{i=0}^{\infty} \Sigma_i$. The maximality and saturation with respect to $\Diamond$ is clear by the construction. The consistency is shown in the standard way by the consistency of Σ_n for $n \geq 1$. Finally, it is clear by the construction that Σ' is saturated with respect to $\Diamond$. □

Definition 4.6. Let W_0 be the set of the maximal consistent sets saturated with respect to $\Diamond$. We define λ_n and H_n $(0 \leq n \leq d(\Sigma))$ as follows:

- Define $H_0 = W_0$ and for each $w \in H_0$, $\lambda_0(w) = w$.
- Let $H_{n+1} = \{hA | h \in H_n \text{ and } \langle A\rangle\top \in \lambda_n(h)\}$. For every $h = h'A \in H_{n+1}$, define $\lambda_{n+1}(h) = \{\phi | \langle A\rangle\phi \in \lambda_n(h')\}$.

We define a function λ in such a way that $\Lambda(\sigma) = \Lambda_n(\sigma)$ for each n.

Propositoin 4.7 (Correctness)**.** For each $n \geq 0$, for each $h \in H_n$, $\lambda_n(h)$ is a maximally consistent set saturated with respect to $\Diamond$.

Proof. The proof of the maximal consistency of $\lambda(h)$ works in the same as in $TPAL$ (in [6]). Saturation with respect to $\Diamond$ of $\lambda(h)$ is guaranteed by the construction, $\lambda(hA) = \{\phi | \langle A\rangle\phi \in \lambda(g)\}$ and the definition of saturation. □

From W_0, we construct the canonical model in the same way as in [6].

Definition 4.8 (Canonical Model)**.** The canonical model of $TAPAL$ is $\mathcal{H}_{can} = (H_{can}, \sim_{can}, V_{can})$, where each element is defined as follows:

- $\mathcal{H}_{can} := \bigcup_{i=0}^{\infty} H_i$.
- $\sim_0$: $\mathcal{A} \mapsto H_0 \times H_0$: for $w, v \in H_0, \sim_{can} (i)(w, v)$ iff $\{\phi | K_i\phi \in a\} \subseteq b$.
- $\sim_{can}$: $\mathcal{A} \mapsto H_{can} \times H_{can}$: for $g, h \in H_{can}$ where $g = w\sigma$ and $h = v\tau$, $\sim_{can} (i)(g, h)$ iff $\sim_0 (i)(w, v)$ and σ, τ are syntactically identical.
- For every $p \in P$, $V_0(p) = \{w \in H_0 | p \in \lambda(w)\}$.
- For every $p \in P$ and $w\sigma \in H_{can}$ with $w \in H_0$, $w\sigma \in V_{can}(p)$ iff $w \in V_0(p)$.

Lemma 4.9. (Truth Lemma) For every formula $\phi \in \mathcal{L}_{TAPAL}$,

$$\phi \in \lambda(h) \quad \text{iff} \quad \mathcal{H}_{can}, h \models \phi.$$

Proof. : The proof is by induction on ϕ. For the cases other than $\Diamond$, the argument is similar to [6]. Assume that ϕ is of the form $\Diamond\psi$. First assume that $\Diamond\psi \in \lambda(h)$. Since $\lambda(h)$ is saturated with respect to $\Diamond$, we have $\langle\theta\rangle\psi \in \lambda(h)$ for some propositional letter θ. By the construction of $\mathcal{H}_{can}$, we have $\psi \in \lambda(h\theta)$. By IH, we obtain $\mathcal{H}_{can}, h\theta \models \psi$. Therefore, we have $\mathcal{H}_{can}, h \models \Diamond\psi$ by truth definition. For the other direction, assume that $\mathcal{H}_{can}, h \models \Diamond\psi$. By definition, there is some θ such that $h\theta \in H_{can}$ and $\mathcal{H}_{can}, h\theta \models \psi$. By IH, we have $\psi \in \lambda(h\theta)$, which, by the construction of $\mathcal{H}_{can}$, implies $\langle\theta\rangle\phi \in \lambda(h)$. This implies by Gen that $\Diamond\phi \in \lambda(h)$. □

Theorem 4.10 (Completeness)**.** $TAPAL$ is weakly complete with respect to $Forest(PAL)$.

Proof. We can show that $\mathcal{H}_{can}$ is generated from $\mathcal{M}_{can} = (W_0, \sim_0, V_0)$ in Definition 4.6 and 4.8, and f_{can} defined by $f_{can}(w) = \{\sigma | w\sigma \in H_{can}\}$ for all $w \in W_0$ in a similar way to [**]. The rest of the argument is standard. □

4.2 $TASPAL$

Next, we give the complete axiomatization of $TASPAL$. Let us denote by $\Box^n$ and $\Diamond^n$ the sequence of n $\Box$'s and *lozenge*'s respectively. Particularly, when $n = 0$, they denote the empty sequence.

Definition 4.11 (Axiomatization)**.** The axiomatization of $TASPAL$ adds to $TASPAL$ the following axioms and inference rule:

Fix $\Diamond^*\phi \leftrightarrow \phi \vee \Diamond\Diamond^*\phi$

$R(\Box^*)$ If $\vdash \phi \rightarrow [\sigma]\Box^n\psi$ for all $n \geq 0$ with σ a sequence of formula, then $\vdash \phi \rightarrow [\sigma]\Box^*\psi$, The role of Fix can be understood by the analogy with the fixed point axiom in PDL (see e.g. [7]). That is,

$$\langle \pi^* \rangle \phi \leftrightarrow \phi \vee \langle \pi \rangle \langle \pi^* \rangle \phi.$$

In the light of this, it would have been more natural if we had an axiom corresponding directly to the Induction Axiom:

Ind $\phi \wedge [\pi^*](\phi \to [\pi]\phi) \to [\pi^*]\phi$

Such a $TASPAL$ axiom would be of the form:

Ind_{TAPAL} $\phi \wedge \Box^*(\phi \to \Box\phi) \to \Box^*\phi$.

Thus, one might wonder where $R(\Box^*)$ comes from instead of **ASInd**. In fact, $R(\Box^*)$ finds a counterpart in an alternative complete axiomatization of PDL (see [14, 17]). That is the axiomatization of PDL with usual axioms except **Ind** and the following rule:

$R([\pi^*])$ If $\vdash \phi \to [\pi'][\pi]^n\psi$ for all $n \geq 0$, then $\vdash \phi \to [\pi'][\pi^*]\psi$.

Theorem 4.12 (Soundness)**.** The axiomatization of $TASPAL$ is sound.

Proof. The soundness of Fix is straighfoward. For $R(\Box^*)$, assume that $\mathcal{H}, h \models \phi \wedge \langle \sigma \rangle \Diamond^* \psi$. Then, we have $\mathcal{H}, h\sigma\tau \models \psi$ for some τ. Thus, $\mathcal{H}, h\sigma \models \Diamond^{len(\tau)}\psi$. Therefore, $\mathcal{H}, h \models \phi \wedge \langle \sigma \rangle \Diamond^{len(\tau)}\psi$. □

Also we can give a completeness proof in the way similar to $TAPAL$. The idea is to construct the canonical model by taking maximally consistent set saturated not only with respect to $\Diamond$ but also with respect to $\Diamond^*$ in the following sense.

Definition 4.13 (Saturation wrt $\Diamond^*$)**.** A set Σ of formulas is *saturated with respect to* $\Diamond^*$, if, for every formula of the form $\langle \sigma \rangle \Diamond^* \phi$ with σ a sequence of formulas, $\langle \sigma \rangle \Diamond^* \phi \in \Sigma$ implies that there is some n such that $\langle \sigma \rangle \Diamond^n \phi \in \Sigma$.

We need the corresponding Lindenbaum's lemma with the two notions of saturation.

Lemma 4.14 (Lindenbaum)**.** Every consistent formula ϕ in $TASPAL$ can be expanded to a maximally consistent set containing ϕ that is saturated with respect to $\Diamond$ and $\Diamond^*$.

Proof. Let $\alpha_1, .\alpha_2 ...$ be an enumeration of the TASPAL-formulas such that $\alpha_1 = \phi$. We construct a sequence $\Sigma_0, \Sigma_1, ...$ of sets as follows:

- $\Sigma_0 = \emptyset$
- If $\Sigma_n \cup \{\alpha_n\}$ is inconsistent, $\Sigma_{n+1} = \Sigma_n$.

- If $\Sigma_n \cup \{\alpha_n\}$ is consistent and α_n is neither of the form $\langle\sigma\rangle\Diamond\psi$ nor of the form $\langle\sigma\rangle\Diamond^*\psi$, $\Sigma_{n+1} = \Sigma_n \cup \{\alpha_n\}$.
- If $\Sigma_n \cup \{\alpha_n\}$ is consistent and α_n is of the form $\langle\sigma\rangle\Diamond\psi$, $\Sigma_{n+1} = \Sigma_n \cup \{\langle\sigma\rangle\Diamond\psi, \langle\sigma\rangle\langle p\rangle\psi\}$ for a propositional letter $p \notin P_\psi \cup P_\sigma \cup P_{\Sigma_n}$. Such a propositional letter exists since Σ_n is finite and we have a countable set of propositional letters.
- If $\Sigma_n \cup \{\alpha_n\}$ is consistent and α_n is of the form $\langle\sigma\rangle\Diamond^*\psi$, take k such that $\Sigma_n \cup \{\langle\sigma\rangle\Diamond^*\psi, \langle\sigma\rangle\Diamond^k\psi\}$ is consistent, and put $\Sigma_{n+1} = \Sigma_n \cup \{\langle\sigma\rangle\Diamond^*\psi, \langle\sigma\rangle\Diamond^k\psi\}$.

We first show that if $\Sigma_n \cup \{\alpha_n\}$ is consistent and α_n is of the form $\langle\sigma\rangle\Diamond^*\psi$, there is always some m such that $\Sigma_n \cup \{\alpha_n, \langle\sigma\rangle\Diamond^m\psi\}$ is consistent. Suppose toward contradiction that there is no such m. Then, for all $m \geq 0$, we have:

$$\vdash \bigwedge \Sigma_n \to \neg\langle\sigma\rangle\Diamond^m\psi.$$

where $\bigwedge \Sigma_n$ is a conjunction of the formulas in Σ_{m-1}. This implies that, for all m,

$$\vdash \bigwedge \Sigma_n \cup \{\alpha_n\} \to [\sigma](\Box^m\neg\psi)$$

and, by $R(\Box^*)$

$$\vdash \bigwedge \Sigma_n \cup \{\alpha_n\} \to [\sigma]\Box^*\neg\psi).$$

Therefore, we have $\Sigma_n \cup \{\alpha_n\} \vdash [\sigma]\Box^*\neg\psi)$ and thus $\Sigma_n \cup \{\alpha_n\} \vdash \neg\langle\sigma\rangle\Diamond^*\psi$. This contradicts our assumption that $\Sigma_n \cup \{\langle\sigma\rangle\Diamond^*\psi\}$ is consistent.

Given this, by following the argument given in the proof of Lemma 4.5, we can show the consistency of Σ_n for all n.

Now we claim that $\Sigma' = \bigcup_{i=0}^{\infty} \Sigma_i$ is the maximal consistent set saturated with respect to $\Diamond$ and $\Diamond^*$. The maximality and saturation with respect to $\Diamond$ and $\Diamond^*$ are clear by construction. Thus, it remains to show that Σ' is consistent. For this, it suffices to show that the formula in each node of the proof for $\Sigma' \vdash \theta$ is in Σ'. The proof is by induction on the depth of deductions. (Note that the proof in $TASPAL$ is finite in depth, but not in width.) Base cases are trivial. For inductive step, assume that premises of a given inference are added at some point of the construction of Σ'. First assume that the conclusions are given by one of the two finitary inference, modus ponens and $R(\Box)$. Then, assume toward contradiction that the conclusion is not in Σ'. In this case, by the maximality of Σ', the negation of the conclusion is in Σ'. However, this implies that there is some i such that Σ_i contains the negation of the conclusion and the premise of the inference. This contradicts the consistency of Σ_i. Now for the infinitary case $R(\Box^*)$ case, assume that the conclusion $\phi \to [\sigma]\Box^*\psi$ is not in Σ'. Then, by maximality, $\phi \wedge \langle\sigma\rangle\Diamond^*\psi)$ is in Σ'. However, then there is some i such that $\langle\sigma\rangle\Diamond^*\neg\psi$ is in Σ_i. Thus, by the construction, there is some k such that $\langle\sigma\rangle\Diamond^k\neg\psi]$ is in Σ_i. Since all the premises of the inferences are in Σ' by IH, it follows that there is some n such that Σ_n contains $\phi \to [\sigma]\Box^k\psi)$ and $\phi \wedge \langle\sigma\rangle\Diamond^l\psi$, which contradicts the consistency of Σ_i. $\Box$

For the Truth Lemma, we construct a canonical model which only consists of maximally consistent sets saturated with respect to $\Diamond$ and $\Diamond^*$. The construction of the canonical model is otherwise the same as in Definition 4.6 and 4.8.

Lemma 4.15 (Correctness)**.** For each $n \geq 0$, for each $\sigma \in H_n$, $\lambda_n(h)$ is a maximally consistent set and saturated with respect to $\Diamond$ and $\Diamond^*$.

Proof. : Similar to the proof of Lemma 4.7. $\square$

We also need the following proposition.

Propositoin 4.16. Let $\sigma \in (\mathcal{L}_{el})^*$ and $len(\sigma) = n$. Then,

1. $\vdash \langle\sigma\rangle\phi \to \Diamond^n\phi$.
2. $\vdash \Diamond^n\phi \to \Diamond^*\phi$.

Proof. First, we prove that $\vdash \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$. By (the contraposition of) Gen and propositional logic, we have $\vdash \Box(\phi \to \psi) \wedge \Box\phi \to [p](\phi \to \psi) \wedge [p]\phi$ for any $p \notin P_\phi \cup P_\psi$. Now by A1, we have $\vdash [p](\phi \to \psi) \wedge [p]\phi \to [p]\psi$. Thus, $\vdash \Box(\phi \to \psi) \wedge \Box\phi \to [p]\psi$. Since $p \notin P_\phi \cup P_\psi$, we obtain by $R(\Box)$ and propositional logic, $\vdash \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$.

Given this, both items are proven by induction on n. For the first claim, the base case is clear. For inductive step, assume the claim holds for n. Then, $\vdash \langle\sigma\rangle\langle\theta\rangle\phi \to \Diamond^n\langle\theta\rangle\phi$. Also by Gen and repeated applications of necessitation for $[p_i]$, we can get $\vdash [p_1]\dots[p_n](\Box\neg\phi \to [\theta]\neg\phi)$ with $p_1, \dots, p_n$ is a sequence of distinct variable that are not in $P_\phi \cup P_\theta$. By repeated applications of $R(\Box)$, we obtain $\vdash \Box^n(\Box\neg\phi \to [\theta]\neg\phi)$ (Note propositionally $\vdash \chi$ iff $\vdash \top \to \chi$). By the above distributive law, this implies $\vdash \Diamond^n\langle\theta\rangle\phi \to \Diamond^{n+1}\phi$. Thus, we have $\vdash \langle\sigma\rangle\langle\theta\rangle\phi \to \Diamond^{n+1}\phi$.

For the second claim, the base case is given by Fix. Then, for inductive step, assume we have $\Box^*\phi \to \Box^n\phi$. By necessitation for $[p]$ and $R(\Box)$ as above, we obtain $\vdash \Box(\Box^*\phi \to \Box^n\phi)$. By standard modal reasoning, we have $\vdash \Box\Box^*\phi \to \Box\Box^n\phi$. From Fix, we have $\vdash \Box^*\phi \to \Box\Box^*\phi$. We have $\vdash \Box\Box^*\phi \to \Box^{n+1}\phi$. $\square$

Lemma 4.17 (Truth Lemma)**.** For every formula $\phi \in \mathcal{L}$,

$$\phi \in \lambda(h) \quad \text{iff} \quad \mathcal{H}_{can}, h \models \phi.$$

Proof. By induction on ϕ. We only do the $\Diamond^*$ case, but the other cases are similar to the proof of Lemma 4.9. Assume that $\Diamond^*\psi \in \lambda(h)$. Since $\lambda(h)$ is a maximally consistent set saturated with respect to $\Diamond^*$, there is some $k \geq 0$ such that $\Diamond^k\psi \in \lambda(h)$ Now, since $\lambda(h)$ is also satuarated with respect to $\Diamond$, we have $\langle\theta_1\rangle \dots \langle\theta_k\rangle\psi \in \lambda(h)$. Thus, by the construction of canonical model, we have $\psi \in \lambda(h\theta_1 \dots \theta_k)$, which implies by IH that $\mathcal{H}_{can}, h\theta_1 \dots \theta_k \models \psi$. This gives us $\mathcal{H}_{can}, \sigma \models \Diamond^*\psi$.

Assume that $\mathcal{H}_{can}, h \models \Diamond^*\psi$. By definition, this is equivalent to saying that there is some σ such that $h\sigma \in \mathcal{H}_{can}$ and $\mathcal{H}_{can}, h\sigma \models \psi$. By IH, we have $\psi \in \lambda(h\sigma)$, which, by the construction of λ, implies $\langle\sigma\rangle\psi \in \lambda(h)$. By Proposition 4.16, we have that $\Diamond^*\psi \in \lambda(h)$. $\square$

Theorem 4.18 (Completeness)**.** $TASPAL$ is weakly complete with respect to $Forest(PAL)$.

Proof. Similar to the proof of Theorem 4.10. □

5 Wider Perspective

Thus, we introduced in the $TPAL$-setting the two generalized operators $\Diamond$ and $\Diamond^*$ to reflect the fact that $TPAL$ distinguishes single announcements and sequences of announcements. We saw that the operators behave in a semantically different way, and that each operator add expressive power to $TPAL$. Also we showed that the extensions of $TPAL$ with the operators find complete axiomatization.

Now the $TPAL$-setting merges ETL and PAL, which is a specific type of DEL that deals only with a particular type of informational events, i.e. the events that eliminates the epistemic possibilities of a given formula. However, DEL can describe a wider range of informational events by *event models* (see e.g. [2, 11, 4]), and $TPAL$ is a part of the story to merge ETL and DEL with full *event models*, as is presented in [6]. Thus, the logic of general event protocol is a topic of further research. Also, the generalized operators for event models in the standard DEL-setting has been proposed in several places ([1, 13]). Thus to consider such a generalized operator in the merged system is also a topic of interest. Particularly, quantification over informational events with a richer description is useful to model various kinds of epistemic concepts, such as evidence, justification, tracking, etc., since those concepts, implicitly or explicitly, involve the type of quantification.

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