Tutorial on Generalized Baire Spaces

Jouko Väänänen

August 23, 2018

What is DST

- Properties of completely arbitrary sets (of reals) are complicated, unintuitive and paradoxical.
- Properties of “definable” sets (of reals) are nice, regular.
- An example: A closed set of reals is either countable or contains a perfect subset. (Cantor-Bendixson Theorem)

Generalized DST

- In the classical Baire space the intuition is (?) measurement in science.
- Generalized Baire space: Is there an intuition?
- Sikorski (1948) proposed an analogue of the reals “higher up”.
- But: reals = unique completely ordered field, so cannot expect a completely ordered field “higher up”.
- However, Scott (1969): Every ordered field has a completion i.e. extension which cannot be further extended with the original field dense. (“Proved 1961, but there were no applications...”)

- One possible intuition comes from uncountable models.
- Why are we interested in the cardinals $\aleph_n$, $n > 0$? Because they are there? Because any of them could be the size of the continuum?
- Why are we interested in uncountable models? Because they are there (LS-theorem!)? Because the field of real numbers is one of them?

- Countable models with countable vocabulary can be thought of as points in the Baire space ω^ω.
- Likewise, models $\mathcal{M}$ of cardinality κ with vocabulary of cardinality κ can be thought of as points in the set κ^κ.
- We can make κ^κ a topological space by letting the sets

$$N(f, \alpha) = \{g \in \kappa^\kappa : f \upharpoonright \alpha = g \upharpoonright \alpha\},$$

  where $\alpha < \kappa$, form the basis of the topology.
- Let us denote this *Generalized Baire Space* κ^κ by $\mathcal{N}_\kappa$.
- Now properties of models of size κ correspond to subsets of $\mathcal{N}_\kappa$. In particular, modulo coding, isomorphism of structures of cardinality κ becomes an "analytic" property in this space.

One of the basic questions about models of size κ that we can try to attack with methods of logic is the question which of those models can be identified up to isomorphism by means of a set of invariants, which is almost the same question as how complicated is the isomorphism relation among (elementarily equivalent) models of the same cardinality.

Shelah's Main Gap Theorem[1] gives one answer: If $\mathcal{M}$ is any structure of cardinality $\kappa \geq \omega_1$ in a countable vocabulary, then the first-order theory of $\mathcal{M}$ is either of the two types:

Structure Case All uncountable models elementary equivalent to $\mathcal{M}$ can be characterized in terms of dimension-like invariants.

Non-structure Case In every uncountable cardinality there are non-isomorphic models elementary equivalent to $\mathcal{M}$ that are *extremely difficult* to distinguish from each other by means of invariants.

Equivalently, we can talk about countable complete first order theories.

[1] For a Crash Course on Shelah's Classification Theory, see "Hirvonen" at http://www.helsinki.fi/sls2015/materials

- The methods we develop in this tutorial can be used to analyze further the non-structure case.
- The above *“extremely difficult to distinguish from each other”* can be analyzed in terms of DST, but have to go “higher up” as the models are uncountable.
- We need GDST !

- A set $A \subseteq \mathcal{N}_\kappa$ is *dense* if A meets every non-empty open set.
- The space $\mathcal{N}_\kappa$ has a dense subset of size $\kappa^{<\kappa}$ consisting of all eventually constant functions.
- If the *Generalized Continuum Hypothesis GCH* is assumed, then $\kappa^{<\kappa} = \kappa$ for all regular κ and $\kappa^{<\kappa} = \kappa^+$ for singular κ.

Baire Category Theorem

Theorem (Scott 1957)

Suppose A_α, $\alpha < \kappa$, are dense open subsets of $\mathcal{N}_\kappa$. Then $\bigcap_\alpha A_\alpha$ is dense.

- Let $f_0 \in \mathcal{N}_\kappa$ and $\alpha_0 < \kappa$ be arbitrary. W.l.o.g. $f_0 \in A_0$.
- If f_ξ and α_ξ for $\xi < \beta$ have been defined so that $\alpha_\zeta < \alpha_\xi$ and $f_\xi \in N(f_\zeta, \alpha_\zeta)$ for $\zeta < \xi < \beta$, then we define f_β and α_β as follows: Choose some $g \in \mathcal{N}_\kappa$ such that $g \in N(f_\xi, \alpha_\xi)$ for all $\xi < \beta$ and let $\alpha_\beta = \sup_{\xi<\beta} \alpha_\xi$.
- Since A_β is dense, there is $f_\beta \in A_\beta \cap N(g, \alpha_\beta)$.
- When all f_ξ and α_ξ for $\xi < \kappa$ have been defined, we let f be such that $f \in N(f_\xi, \alpha_\xi)$ for all $\xi < \kappa$. Then $f \in \bigcap_\alpha A_\alpha \cap N(f_0, \alpha_0)$.

A sanity test

Definition

A set is nowhere dense if its closure has empty interior. Any union of $\leq \kappa$ nowhere dense subsets of $\mathcal{N}_\kappa$ is called a set of the first category, otherwise of the second category.

Note that singletons are nowhere dense and therefore sets of cardinality $< \kappa$ are necessarily of the first category.

A consequence of the Baire Category Theorem:

Corollary

$\mathcal{N}_\kappa$ is of the second category.

Cantor-Bendixson Theorem

- The Cantor-Bendixson Theorem for $\mathcal{N}_\kappa$ would say: Every closed subset of $\mathcal{N}_\kappa$ is either of cardinality $\leq \kappa$ or contains a perfect subset.
- A set is perfect if it is closed and has no isolated points.
- What is a "perfect" subset of $\mathcal{N}_\kappa$? Intuitively, a perfect subset of $\mathcal{N}_\omega$ is a closed set which splits everywhere, so you can form a full binary tree inside it.
- If we take a closed subset of $\mathcal{N}_{\aleph_1}$ which splits everywhere, we cannot necessarily form a full binary tree of height ω_1 because we do no know how to go over limits.
- As in infinitary languages, the answer is we need a strategy for limits.

Perfectness

Definition (V. 91)

A set $A \subseteq \mathcal{N}_\kappa$ is ω_1-perfect if for all $f \in A$ player II has a winning strategy in the game $G(A, f)$ in which on move α player I chooses a countable ordinal ξ_α and then II chooses $f_\alpha \in A \setminus \{f\}$ such that $f_\alpha \neq f_\beta$ and $f_\alpha \restriction \xi_\beta = f_\beta \restriction \xi_\beta$ for all $\beta < \alpha$. Player II wins if she can play all ω_1 moves.

I	ξ_0		ξ_1		$\dots$	ξ_α		$\dots$
II		f_0		f_1	$\dots$		f_α	$\dots$

If ω_1 is replaced by ω, we get the usual concept of perfectness. If player I wins for all $f \in A$, the set is called ω_1-scattered.

Lemma

Every non-empty closed ω_1 *-perfect subset of* $\mathcal{N}_\kappa$ *has cardinality at least* 2^{ω_1}. *QED*

The natural form of the Cantor-Bendixson Theorem for $\mathcal{N}_{\omega_1}$ is now: Every closed subset of $\mathcal{N}_{\omega_1}$ of cardinality $> \aleph_1$ contains an ω_1-perfect subset.
But this is not necessarily true. Even the following weaker statement need not be true: Every closed subset of $\mathcal{N}_{\omega_1}$ of cardinality $> \aleph_1$ is of cardinality 2^{ω_1}.

Definition

$I(\omega)$ is the statement that there is a normal ω_2-complete ideal I on ω_2 such that I^+ contains a dense σ-closed subset.

$I(\omega)$ is equiconsistent with a measurable cardinal. $I(\omega)$ implies CH.

Lemma (V. 91)

Suppose $A \subseteq \mathcal{N}_{\omega_1}$.

1. *If A is of cardinality $\leq \aleph_1$ and closed, then it is ω_1-scattered.*
2. *If A is of cardinality $> \aleph_1$ and $I(\omega)$ holds, then it contains an ω_1-perfect subset.*

Proof.

(1): Diagonalization. (2): W.l.o.g. $|A| = \aleph_2$. Let I be as in $I(\omega)$ on A with σ-closed dense $K \subseteq I^+$. We call f an I-point of A if every neighbourhood of f meets A in an I-positive set. Let A' be the set of I-points of A. The strategy of II is to play side-moves $A_\alpha \in K$, a descending set of I-points of A, all moves she could make. □

$G(A, f)$ with side-moves for ***II***.

I	ξ_0		ξ_1		$\dots$	ξ_α		$\dots$
II		f_0, A_0		f_1, A_1	$\dots$		f_α, A_α	$\dots$

Theorem (V. 91)

Assume $I(\omega)$. Suppose $A \subseteq \mathcal{N}_{\omega_1}$ is closed. Let

$$K = \{f \in A : \textit{II has a winning strategy in } G(A, f)\}$$

$$S = \{f \in A : \textit{I has a winning strategy in } G(A, f)\}.$$

Then

1. $K \cap S = \emptyset$, $K \cup S = A$.
2. $|S| \leq \aleph_1$, $|K| \in \{0, 2^{\aleph_1}\}$
3. *S is ω_1-scattered (the ω_1-scattered part).*
4. *K is ω_1-perfect and closed (the ω_1-kernel).*

Proof.

Suppose $f \notin S$. We show $f \in K$. The strategy of ***II*** is to use side-moves in the the dense σ-closed set of I-points, as above. □

- In Generalized Descriptive Set Theory it is not enough to generalize the space. We *have to* generalize the concept of ordinal as well.
- We can think of ordinals as trees (of any size) without infinite branches.
- Our generalization of an ordinal is a tree (of any size) without a branch of length κ.

- Let $\mathcal{T}_\kappa$ be the class of trees without κ-branches, and $\mathcal{T}_{\lambda,\kappa}$ the class of trees in $\mathcal{T}_\kappa$ of cardinality $\leq \lambda$.
- The elements of $\mathcal{T}_\kappa$ are our generalized ordinals.
- For any tree T we denote by σT the tree of ascending chains of T ordered by end-extension (Kurepa, see later).
- σT is like the “successor” of T.

Definition
Suppose $\mathcal{P}$ and $\mathcal{P}'$ are po-sets. We define

$$\mathcal{P} \leq \mathcal{P}'$$

if there is a mapping $f : P \to P'$ such that for all $x, y \in P$:

$$x <_{\mathcal{P}} y \to f(x) <_{\mathcal{P}'} f(y).$$

We write $\mathcal{P} < \mathcal{P}'$, if $\mathcal{P} \leq \mathcal{P}'$ and $\mathcal{P}' \nleq \mathcal{P}$, and we write $\mathcal{P} \equiv \mathcal{P}'$, if $\mathcal{P} \leq \mathcal{P}'$ and $\mathcal{P}' \leq \mathcal{P}$.

Lemma
If $\mathcal{P} \leq \mathcal{P}'$ and $\mathcal{P}$ has a κ-branch, then so does $\mathcal{P}'$. QED

- Note that $\leq$ is a transitive relation among po-sets.
- The $\equiv$-classes of $\leq$ form a quasi-ordered class.
- It is not a total order, for there are incomparable po-sets, for example $(\omega, <)$ and its inverse ordering $(\omega, >)$.
- For simplicity, we call $\leq$ itself the quasi-order of po-sets, without recourse to the $\equiv$-classes.

Example

For any ordinal α let B_α be the tree of descending sequences $\beta_0 > \ldots > \beta_n$ of elements of α ordered by end-extension. It is easy to show that $\alpha \leq \beta$ (as ordinals) if and only if $B_\alpha \leq B_\beta$ as po-sets. Every well-founded tree is $\equiv$-equivalent to some B_α.

Let us modify the game $G(A, f)$ to $G(A, f, T)$ by asking ***I*** to make side-moves in a tree T, going up the tree move by move:

Definition

I	ξ_0, t_0		ξ_1, t_1		$\dots$	ξ_α, t_α		$\dots$
II		f_0		f_1	$\dots$		f_α	$\dots$

Let

$$K_T = \{f \in A : \textit{II} \text{ has a winning strategy in } G(A, f, T)\}$$

$$S_T = \{f \in A : \textit{I} \text{ has a winning strategy in } G(A, f, T)\}.$$

$$T \leq T' \Rightarrow K_T \supseteq K_{T'}$$

$$T \leq T' \Rightarrow S_T \subseteq S_{T'}$$

Theorem

Suppose $A \subseteq \mathcal{N}_{\omega_1}$ is closed with ω_1-kernel K and ω_1-scattered part S. Then $K = \bigcap_{T \in \mathcal{T}_\kappa} K_T$, $S = \bigcup_{T \in \mathcal{T}_\kappa} S_T$.

Proof.

Of course, $K \subseteq K_T$. Suppose $f \notin K$ i.e. *II* does not have a winning strategy in $G(A, f)$. Let T be the tree of short winning strategies[2] of ***II***, i.e. in games shorter than κ, ordered by end-extensions of the strategies. Note that $T \in \mathcal{T}_{\omega_1}$, hence also $\sigma T \in \mathcal{T}_{\omega_1}$. Now $f \notin K_{\sigma T}$, for otherwise $f \in K$. Similarly, $S_T \subseteq S$ trivially. Let then $f \in S$ i.e. I wins with some strategy τ. Let T be the tree of sequences of moves where I has used τ. Now $f \in S_{\sigma T}$. □

[2] An idea from Hyttinen 87.

Analytic sets

Definition

A subset *A* of $\mathcal{N}_\kappa$ is said to be $\mathbf{\Sigma}^1_1$ (or *analytic*) if it is a projection of a closed subset of $\mathcal{N}_\kappa \times \mathcal{N}_\kappa$. A set is $\mathbf{\Pi}^1_1$ (or *co-analytic*) if its complement is analytic. Finally, a set is $\mathbf{\Delta}^1_1$ if it is both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$.

Example

Examples of analytic sets relevant if κ is a regular cardinal $> \omega$, are

$$\mathrm{CUB}_\kappa = \{f \in \mathcal{N}_\kappa : \{\alpha < \kappa : f(\alpha) = 0\} \text{ contains a club}\}$$

and

$$\mathrm{NS}_\kappa = \{f \in \mathcal{N}_\kappa : \{\alpha < \kappa : f(\alpha) \neq 0\} \text{ contains a club}\}.$$

Definition

The class of *Borel* subsets of $\mathcal{N}_\kappa$ is the smallest class containing the open sets and the closed sets, and which is closed under unions and intersections of length κ.

Theorem (Shelah-V. 2000)

Assume $\kappa^{<\kappa} = \kappa > \omega$. *Then* CUB_κ *and* NS_κ *are two disjoint analytic sets that cannot be separated by a Borel set.*

- The set of α-sequences of elements of κ for various $\alpha < \kappa$ form a tree $\mathcal{N}_{<\kappa}$ under the subsequence relation.
- Any subset T of $\mathcal{N}_{<\kappa}$ which is closed under subsequences is called a *tree* here.
- Notation: We denote $\langle g(\beta) : \beta < \alpha \rangle$ by $\bar{g}(\alpha)$.

Lemma (MV 93[3])

A set $A \subseteq \mathcal{N}_\kappa$ *is analytic iff there is a tree* $T \subseteq \mathcal{N}_{<\kappa} \times \mathcal{N}_{<\kappa}$ *such that for all* f*:*

$$f \in A \iff T(f) \text{ has a } \kappa\text{-branch}, \tag{1}$$

where $T(f) = \{\bar{g}(\alpha) : (\bar{g}(\alpha), \bar{f}(\alpha)) \in T\}$. *Such a tree is called a* tree representation *of* A.

[3] Mekler-Väänänen 93

Proof.

Suppose first A is analytic and $B \subseteq \kappa^\kappa \times \kappa^\kappa$ is a closed set such that

$$f \in A \iff \exists g((f, g) \in B).$$

Let

$$T = \{(\bar{f}(\alpha), \bar{g}(\alpha)) : (f, g) \in B, \alpha < \kappa\}.$$

Clearly now $f \in A$ if and only if $T(f)$ has a κ-branch. Conversely, suppose such a T exists. Let B be the set of (f, g) such that $(\bar{f}(\alpha), \bar{g}(\alpha)) \in T$ for all $\alpha < \kappa$. The set B is closed and its projection is A. □

Respectively, a set is co-analytic if and only if there is a tree $T \subseteq \mathcal{N}_{<\kappa} \times \mathcal{N}_{<\kappa}$ such that for all f:

$$f \in A \iff T(f) \text{ has no } \kappa\text{-branches.} \tag{2}$$

Proposition (MV 93)

Suppose B is a co-analytic subset of $\mathcal{N}_\kappa$ *and T is such that* $f \in B \iff T(f)$ *has no* κ*-branches. For any tree* $S \in \mathcal{T}_\kappa$ *let*

$$B_S = \{f \in B : T(f) \leq S\}.$$

Then

$$B = \bigcup_{S \in \mathcal{T}_{\lambda,\kappa}} B_S,$$

where $\lambda = \kappa^{<\kappa}$.

Proof.

Clearly $B_S \subseteq B$ if $S \in \mathcal{T}_\kappa$. Conversely, suppose $f \in B$. Then of course $f \in B_{T(f)}$. It remains to observe that $|T(f)| \leq \kappa^{<\kappa}$. □

Suppose again B is co-analytic and

$$f \in B \iff T(f) \text{ has no } \kappa\text{-branches.}$$

Suppose $A \subseteq B$ is analytic and S is a tree such that

$$f \in A \iff S(f) \text{ has a } \kappa\text{-branch.}$$

Let

$$T' = \{(\bar{f}(\alpha), \bar{g}(\alpha), \bar{h}(\alpha)) : \bar{g}(\alpha) \in T(f), \bar{h}(\alpha) \in S(f)\}. \qquad (3)$$

Note that $|T'| \leq \kappa^{<\kappa}$ and T' has no κ-branches, for such a branch would give rise to a triple (f, g, h) which would satisfy $f \in A \setminus B$. Note also that if $f \in A$, then there is a κ-branch $\{\bar{h}(\alpha) : \alpha < \kappa\}$ in $S(f)$, and hence the mapping $\bar{g}(\alpha) \mapsto (\bar{f}(\alpha), \bar{g}(\alpha), \bar{h}(\alpha))$ witnesses $T(f) \leq T'$.

Covering Theorem for $\mathcal{N}_\kappa$

We have proved:

Proposition (MV 93)

Suppose B is a co-analytic subset of $\mathcal{N}_\kappa$ and S is as in (2). Suppose $A \subseteq B$ is analytic. Then

$$A \subseteq B_T$$

for some $T \in \mathcal{T}_{\lambda,\kappa}$, where $\lambda = \kappa^{<\kappa}$.

The idea is that the sets B_T, $T \in \mathcal{T}_{\lambda,\kappa}$, cover the co-analytic set B completely, and moreover any analytic subset of B can be already covered by a single B_T. Especially if B happens to be $\mathbf{\Delta}^1_1$, then there is $T \in \mathcal{T}_{\lambda,\kappa}$ such that $B = B_T$.

Souslin–Kleene Theorem for $\mathcal{N}_\kappa$

Corollary (MV 93)

Suppose B is a $\mathbf{\Delta}^1_1$ subset of $\mathcal{N}_\kappa$. Then

$$B = B_T$$

for some $T \in \mathcal{T}_{\lambda,\kappa}$, where $\lambda = \kappa^{<\kappa}$.

Luzin Separation Theorem for $\mathcal{N}_\kappa$:

Corollary (MV 93)

Suppose A and B are disjoint analytic subsets of $\mathcal{N}_\kappa$. Then there is a set of the form C_T for some co-analytic set C and some $T \in \mathcal{T}_{\lambda,\kappa}$, where $\lambda = \kappa^{<\kappa}$, that separates A and B, i.e. $A \subseteq C_T$ and $C_T \cap B = \emptyset$.

In the case of classical descriptive set theory, which corresponds to assuming $\kappa = \omega$, the sets B_T are Borel sets. If we assume CH, then CUB_{ω_1} and NS_{ω_1} cannot be separated by a Borel set.

Proposition (MV 93)

If $\kappa^{<\kappa} = \kappa$, then the sets B_T are analytic. If in addition T is a strong bottleneck[4]*, then B_T is $\mathbf{\Delta}^1_1$.*

[4] $\forall T'(T \leq T' \vee T' \leq T) + \epsilon$

Note that $\mathcal{T}_{\kappa,\kappa}$ is essentially a co-analytic subset of $\mathcal{N}_\kappa$, like the set *WO* of reals that code a well-ordering is in the case $\kappa = \omega$. We know that every analytic subset of *WO* has a sup in *WO*. Likewise:

Corollary (MV 93)

Suppose $\kappa^{<\kappa} = \kappa$. Every analytic set of trees in $\mathcal{T}_{\kappa,\kappa}$ has an upper $\leq$-bound in $\mathcal{T}_{\kappa,\kappa}$. (Boundedness Theorem)

Would be trivial for $\mathcal{T}_\kappa$.

Definition

Suppose T is a tree in $\mathcal{T}_\kappa$. A labelling of T is a function L which assigns every maximal branch of T with a basic open set of $\mathcal{N}_\kappa$. Consider the following game played inside T: $G(T, L, f)$:

I	$\bar{g}(1)$		$\bar{g}(3)$		$\ldots$	$\bar{g}(\alpha)$		$\ldots$
II		$\bar{g}(2)$		$\bar{g}(4)$	$\ldots$		$\bar{g}(\alpha+1)$	$\ldots$

Eventually a maximal $\bar{g}(\alpha)$ is obtained. ***I*** wins if $f \in L(\bar{g}(\alpha))$. Otherwise ***II*** wins. Let

$$B(T, L) = \{f \in \mathcal{N}_\kappa : \textbf{\textit{I}} \text{ wins } G(T, L, f)\}.$$

Sets of the form $B(T, L)$ are called Borel*. The set $B(T, L)$ is determined Borel* if the game $G(T, L, f)$ is determined for all f.

Lemma (MV 93)

1. *For $\kappa = \omega$ we get the classical Borel sets (Blackwell 81).*
2. *Borel sets in the usual sense are Borel*.*
3. *Borel*-sets are closed under unions and intersections of length $\leq \kappa$.*
4. *Assuming $\kappa^{<\kappa} = \kappa$, every Borel*-set is $\mathbf{\Sigma}^1_1$.*
5. *The sets B_T are Borel*.*

Corollary (Souslin-Kleene Theorem for $\mathcal{N}_\kappa$)

Assume $\kappa^{<\kappa} = \kappa$. A subset of $\mathcal{N}_\kappa$ is $\mathbf{\Delta}^1_1$ if and only if it is a determined Borel set.*

Definition (Kurepa)

Suppose $\mathcal{P}$ is a po-set. The tree $\sigma\mathcal{P}$ is defined as follows. Its domain is the set of functions s with $\mathrm{dom}(s) \in \mathrm{On}$ such that for all $\alpha, \beta \in \mathrm{dom}(s)$

$$\alpha < \beta \to s(\alpha) <_{\mathcal{P}} s(\beta).$$

The order is

$$s \leq s' \leftrightarrow s = s' \restriction_{\mathrm{dom}(s)}.$$

$\sigma'\mathcal{P}$ is the suborder of $\sigma\mathcal{P}$ consisting of sequences $s \in \sigma\mathcal{P}$ of successor length.

Lemma (Kurepa)

(i) $\sigma'\mathcal{P} \leq \mathcal{P}$.

(ii) $\sigma\mathcal{P} \not\leq \mathcal{P}$.

(iii) $\sigma'\mathcal{P} < \sigma\mathcal{P}$.

(iv) *If T is a tree, then $T \equiv \sigma' T$.*

proof

(i) If $s \in \sigma'\mathcal{P}$, let $f(s) = s(\mathrm{dom}(s) - 1)$. Then $f : \sigma'\mathcal{P} \to \mathcal{P}$ is order-preserving.
(ii) Suppose $f : \sigma\mathcal{P} \to \mathcal{P}$ were order-preserving. Define inductively $s : \mathrm{On} \to \mathcal{P}$ by $s(\alpha) = f(s\restriction_\alpha)$. Since $\alpha < \beta$ implies $s(\alpha) <_{\mathcal{P}} s(\beta)$, we get the result that $\mathcal{P}$ is a proper class, a contradiction.
(iii) $\sigma'\mathcal{P} \leq \sigma\mathcal{P}$ trivially. On the other hand, if $\sigma\mathcal{P} \leq \sigma'\mathcal{P}$, then $\sigma\mathcal{P} \leq \mathcal{P}$ contrary to (ii),
(iv) We already know $\sigma' T \leq T$. Suppose $t \in T$ and $\langle t_\alpha : \alpha \leq \beta\rangle$ is the set of $t' \in T$ with $t' \leq_T t$ in ascending order. Let $\mathrm{dom}(s) = \beta + 1$ and $s_t(\alpha) = t_\alpha$. Then $s_t \in \sigma' T$ and $t \mapsto s_t$ is order-preserving. proof

Lemma (HV 90[5])

There is no sequence $\mathcal{P}_0, \mathcal{P}_1, \ldots$ so that $\sigma\mathcal{P}_{n+1} \leq \mathcal{P}_n$ for all $n < \omega$.

Proof.

Suppose $f_n : \sigma\mathcal{P}_{n+1} \to \mathcal{P}_n$ is order-preserving. For each fixed α, let $s^n_\alpha \in \mathcal{P}_n$ so that $f_n(\langle s^{n+1}_\beta : \beta < \alpha \rangle) = s^n_\alpha$. Then each $\mathcal{P}_n$ is a proper class, a contradiction. □

Definition (HV 90)

Define $\mathcal{P} \ll \mathcal{P}' \iff \sigma\mathcal{P} \leq \mathcal{P}'$.

Corollary (HV 90)

$\ll$ *is a well-founded relation (although the $\mathcal{P}$ need not be).*

---

[5] Hyttinen-Väänänen 90

Definition (HV 90)

Suppose $\mathcal{P}$ and $\mathcal{P}'$ are po-sets. The game $G(\mathcal{P}, \mathcal{P}')$ is defined as follows. Player ***I*** plays $p_0 \in \mathcal{P}$, then player ***II*** plays $p'_0 \in \mathcal{P}'$. After this ***I*** plays $p_1 \in \mathcal{P}$ with $p_0 <_{\mathcal{P}} p_1$, and then player ***II*** plays $p'_1 \in \mathcal{P}'$ with $p'_0 <_{\mathcal{P}'} p'_1$, and so on. At limits player ***I*** moves first $p_\nu \in \mathcal{P}$ with $p_\alpha <_{\mathcal{P}} p_\nu$ for all $\alpha < \nu$. Then ***II*** moves $p'_\nu \in \mathcal{P}'$ with $p'_\alpha <_{\mathcal{P}} p'_\nu$ for all $\alpha < \nu$. If a player cannot move, he loses and the other player wins. Since $\mathcal{P}$ and $\mathcal{P}'$ are sets, one of the players eventually wins.

Lemma (HV 90)

(i) $\sigma'\mathcal{P} \leq \mathcal{P}'$ *if and only if* ***II*** *wins* $G(\mathcal{P}, \mathcal{P}')$.

(ii) *If* $\mathcal{P}$ *is a tree, then* $\mathcal{P} \leq \mathcal{P}'$ *if and only if* ***II*** *wins* $G(\mathcal{P}, \mathcal{P}')$.

Proof.

(i) Suppose $f : \sigma'\mathcal{P} \to \mathcal{P}'$ is order-preserving. If ***I*** has played $p_0 < \ldots < p_\alpha$ in $G(\mathcal{P}, \mathcal{P}')$, ***II*** plays $p'_\alpha = f((p_0, \ldots, p_\alpha))$. In this way she ends up the winner. Conversely, suppose ***II*** wins $G(\mathcal{P}, \mathcal{P}')$ and $s \in \sigma'\mathcal{P}$ with $\mathrm{dom}(s) = \alpha + 1$. Let us play $G(\mathcal{P}, \mathcal{P}')$ so that ***I*** plays $p_\beta = s(\beta)$ for $\beta \leq \alpha$ and ***II*** uses her winning strategy. After ***I*** plays p_α, ***II*** plays p'_α. If we define $f(s) = p'_\alpha$, we get an order-preserving mapping $\sigma'\mathcal{P} \to \mathcal{P}$. This ends the proof of (i). (ii) follows from (i) and the above Lemma (iv). □

Lemma (HV 90)

*$\mathcal{P}' \ll \mathcal{P}$ if and only if **I** wins $G(\mathcal{P}, \mathcal{P}')$.*

Proof.

Suppose $f : \sigma\mathcal{P}' \to \mathcal{P}$ is order-preserving. If ***II*** has played

$$p'_0 < \ldots < p'_\beta < \ldots \ (\beta < \alpha) \tag{4}$$

in $G(\mathcal{P}, \mathcal{P}')$, ***I*** plays $p_\alpha = f((p'_0, \ldots, p'_\beta, \ldots))$ in $\mathcal{P}'$. In this way ***I*** wins $G(\mathcal{P}, \mathcal{P}')$. On the other hand, if ***I*** wins $G(\mathcal{P}, \mathcal{P}')$ and (4) is an ascending chain in $\mathcal{P}'$, we can let ***I*** play against the moves $p'_0, \ldots, p'_\beta, \ldots$ of ***II*** in $G(\mathcal{P}, \mathcal{P}')$. Finally ***I*** plays p_α according to his winning strategy. We let

$$f((p'_0, \ldots, p'_\beta, \ldots)) = p_\alpha.$$

Now $f : \sigma\mathcal{P}' \to \mathcal{P}$ is order-preserving. □

Example (Todorcevic 81)

Suppose $S \subseteq \omega_1$. Let $T(S)$ be the tree of closed ascending sequences of elements of S. Choose disjoint stationary sets S_1 and S_2. Then $T(S_1) \not\leq T(S_2)$ and $T(S_2) \not\leq T(S_1)$. Thus the game $G(T(S_1), T(S_2))$ is non-determined.

Towards an analysis of a particular analytic relation: isomorphism

- We introduce a "clocked" version of the Ehrenfeucht–Fraïssé Game.
- In this game player ***I*** makes moves not only in the models in question but also side-moves in a po-set, going up move by move.
- The game goes on as long as ***I*** can move.

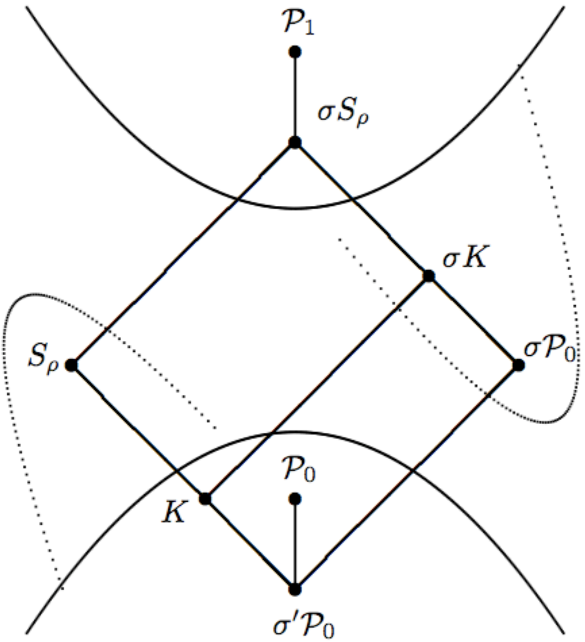
Notation: $\mathrm{b}(\mathcal{P})$ = the least ordinal δ so that $\mathcal{P}$ does not have an ascending chain of length δ.

Definition (Karttunen 84, HV 90)

Suppose $\mathcal{A}_0$ and $\mathcal{A}_1$ are L-structures and $\mathcal{P}$ is a po-set. The *Clocked Ehrenfeucht–Fraïssé Game* $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ is as follows:

- On each round ***I*** chooses $c_\alpha \in \{0, 1\}$, $x_\alpha \in A_{c_\alpha}$, and $p_\alpha \in \mathcal{P}$, and the ***II*** chooses $y_\alpha \in A_{1-c_\alpha}$.
- It is required that $p_0 <_{\mathcal{P}} \ldots <_{\mathcal{P}} p_\alpha <_{\mathcal{P}} \ldots$.
- Finally ***I*** cannot play!
- If during the game a partial isomorphism between $\mathcal{A}_0$ and $\mathcal{A}_1$ is generated, ***II*** has won the game, otherwise ***I*** has won.

- ***I*** wishes $\mathcal{P}$ was bigger (in $\leq$).
- ***II*** wishes $\mathcal{P}$ was smaller (in $\leq$).
- If ***II*** wins against $\mathcal{P}$ and ***I*** wins with $\mathcal{P}$, then $\mathcal{P} \ll \mathcal{P}'$.



$\mathcal{P}_1$
σS_ρ
σK
S_ρ
$\sigma \mathcal{P}_0$
$\mathcal{P}_0$
K
$\sigma' \mathcal{P}_0$

Hyttinen-Tuuri 91

- Unstable theories have models with the lower boundary high.
- Consistently, unsuperstable theories have models with upper boundary high.

Definition (Hyttinen-Tuuri 91)

A tree T is a *universal Scott tree* of a structure $\mathcal{M}$ of cardinality κ if T has no branches of length κ and player ***I*** wins $\mathrm{EF}_{\sigma T}(\mathcal{M}, \mathcal{N})$ whenever $\mathcal{M} \not\cong \mathcal{N}$ and $|N| = |M|$.

*To wipe the smile off **I**'s face we ask can we have $|T| \leq \kappa$?*

Proposition (MV 93)

Suppose $\kappa^{<\kappa} = \kappa$ and $\mathcal{M}$ is a structure with M $= \kappa$. The following are equivalent:

(1) *The orbit of $\mathcal{M}$ is $\mathbf{\Delta}^1_1$.*

(2) *$\mathcal{M}$ has a universal Scott tree of cardinality κ.*

Proof is based on the Souslin-Kleene Theorem for $\mathcal{N}_\kappa$.

- A candidate for *"extremely hard to distinguish from each other"*: Orbit not $\mathbf{\Delta}_1^1$.
- When does a complete first order theory T (in a countable vocabulary) have models in an uncountable cardinality the orbit of which is not $\mathbf{\Delta}_1^1$?
- This question is highly connected to stability-theoretic properties of T.
- Follow the lecture of Hyttinen for more on this.

Summary—what have we learned?

- GDST is a meaningful and rich area of set theory.
- To get direction and intuition one can keep model theory of uncountable models (and stability theory) in mind, as a guideline.
- We can find the topological content of (some) model theory of uncountable models.
- Passing from DST to GDST (not unexpectedly) new challenges emerge, most notably the possible difference between $\kappa^{<\kappa}$ and κ.
- We should isolate scenarios (beyond ZFC) where GDST yields the best results.

Open question

1. Develop GDST when $\kappa^{<\kappa} > \kappa$.
2. Develop GDST of definable sets.
3. Greatest Tree Problem in $\mathcal{T}_{\omega_1}$.
4. Bottleneck Problem in $\mathcal{T}_{\omega_1}$.

[1] Taneli Huuskonen. Comparing notions of similarity for uncountable models. *J. Symbolic Logic*, 60(4):1153–1167, 1995.

[2] Tapani Hyttinen. Games and infinitary languages. *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes*, (64):32, 1987.

[3] Tapani Hyttinen. Model theory for infinite quantifier languages. *Fund. Math.*, 134(2):125–142, 1990.

[4] Tapani Hyttinen and Heikki Tuuri. Constructing strongly equivalent nonisomorphic models for unstable theories. *Ann. Pure Appl. Logic*, 52(3):203–248, 1991.

[5] Tapani Hyttinen and Jouko Väänänen. On Scott and Karp trees of uncountable models. *J. Symbolic Logic*, 55(3):897–908, 1990.

[6] Maaret Karttunen. Model theory for infinitely deep languages. *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes*, (50):96, 1984.

[7] Alan Mekler and Jouko Väänänen. Trees and Π^1_1-subsets of ${}^{\omega_1}\omega_1$. *J. Symbolic Logic*, 58(3):1052–1070, 1993.

[8] Dana Scott. The independence of certain distributive laws in Boolean algebras. *Trans. Amer. Math. Soc.*, 84:258–261, 1957.

[9] Saharon Shelah and Jouko Väänänen. Stationary sets and infinitary logic. *J. Symbolic Logic*, 65(3):1311–1320, 2000.

[10] Roman Sikorski. On an ordered algebraic field. *Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys.*, 41:69–96 (1950), 1948.

[11] Roman Sikorski. Remarks on some topological spaces of high power. *Fund. Math.*, 37:125–136, 1950.

[12] Stevo Todorcevic. Stationary sets, trees and continuums. *Publ. Inst. Math. (Beograd) (N.S.)*, 29(43):249–262, 1981.

[13] Stevo Todorcevic and Jouko Väänänen. Trees and Ehrenfeucht-Fraïssé games. *Ann. Pure Appl. Logic*, 100(1-3):69–97, 1999.

[14] Heikki Tuuri. *Infinitary languages and Ehrenfeucht-Fraisse games*. 1990. Thesis (Ph.D.)–University of Helsinki (Finland).

[15] Jouko Väänänen. A Cantor-Bendixson theorem for the space $\omega_1^{\omega_1}$. *Fund. Math.*, 137(3):187–199, 1991.