CLASSIFICATION THEORY

IN GENERALIZED BAIRE SPACES

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Standing assumption: $\kappa^{<\kappa} = \kappa > \omega$. In addition, some results below require additional cardinal assumptions. We will not mention these unless they are important for this talk. Also by a model we mean a model of size κ and a theory means a countable complete theory.

Why generalized Baire spaces are interesting from the point of view of model theory?

(Vaught) Suppose $X \subseteq \kappa^\kappa$ is closed under isomorphisms. Then X is Borel iff X is $L_{\kappa^+,\kappa}$-definable.

A new way to look the classification problem.

How to classify theories according to their complexity?
What the complexity of a theory mean?

Model theoretic view: A theory is complicated if its models are. Then what makes models complicated?

S. Shelah suggested the following two answers:

Count the number of models.

Works rather well for theories that are classifiable and shallow

but

does not see difference above these.
Almost all theories have the maximal number of models in almost all cardinalities.
Why having many models makes the models complicated?

Ask if $L_{\infty\kappa}$-equivalence imply isomorphism and if it does, ask what is the quantifier rank needed for this.

More intuitive answer for the complexity of models and works rather well for theories that are classifiable

but

does not see any difference above these.

Are there differences in the complexity above classifiable?
Are there other ways to measure the complexity?

Closed games: For all $\alpha < \kappa$, let $A_\alpha \subseteq \kappa^\alpha$. $G((A_\alpha)_{\alpha<\kappa})$ is a game of two players, I and II. In turns the players choose elements $\gamma_i \in \kappa$, $i < \kappa$. II wins if for all $\alpha < \kappa$, $(\gamma_i)_{i<\alpha} \in A_\alpha$. If T is an ∞,κ-tree, the approximation $G_T((A_\alpha)_{\alpha<\kappa})$ is played as before except that at each move, in addition to γ_i, I chooses also $t_i \in T$ bigger than all his earlier choices from T. The game ends when I can not move any more, say at move β, and then II wins if for all $\alpha \leq \beta$, $(\gamma_i)_{i<\alpha} \in A_\alpha$.

I$\uparrow G((A_\alpha)_{\alpha<\kappa})$ iff I$\uparrow G_T((A_\alpha)_{\alpha<\kappa})$ for some κ^+,κ-tree T.

II$\uparrow G((A_\alpha)_{\alpha<\kappa})$ iff II$\uparrow G_T((A_\alpha)_{\alpha<\kappa})$ for all $(2^\kappa)^+,\kappa$-trees T.

$\mathcal{A} \cong \mathcal{B}$ iff II$\uparrow EF_\kappa^\kappa(\mathcal{A},\mathcal{B})$ iff II$\uparrow EF_T^\kappa(\mathcal{A},\mathcal{B})$ for all κ^+,κ-trees T.

(H and Tuuri) If Σ is (e.g.) unstable, then there is $M^* \models \Sigma$ for which for all κ^+, κ-trees T there is $N_T \models \Sigma$ such that $M^* \not\cong N_T$ and II$\uparrow EF^\kappa_T(M^*, N_T)$.

(H and Shelah) Suppose $\kappa = \lambda^+$ and $\lambda^\omega = \lambda = cf(\lambda) > \omega$ and Σ is unsuperstable. Then there are $M, N \models \Sigma$ such that $M \not\cong N$ and II$\uparrow EF^\kappa_{\lambda\times\omega}(M, N)$.

Let $\Sigma_\omega = Th((\omega^\omega, E_n)_{n<\omega})$, where $\eta E_n \xi$ if $\eta \restriction (n+1) = \xi \restriction (n+1)$.

(H and Tuuri, H and Shelah) Suppose $\kappa = \lambda^+ \in I[\kappa]$. Let $M, N \models \Sigma_\omega$. If II$\uparrow EF^\kappa_{\lambda\times\omega+2}(M, N)$, then $M \cong N$.

(Shelah) If Σ is classifiable, then for $M, N \models \Sigma$, $M \cong N$ iff II$\uparrow EF^\kappa_\omega(M, N)$.

(Shelah) If Σ is classifiable and shallow, then there is a κ^+, ω-tree T such that for $M, N \models \Sigma$, $M \cong N$ iff II$\uparrow$ $EF^\kappa_T(M, N)$. And the size (rank) of T depends on the depth of Σ.

So this works a bit better than $L_{\infty,\kappa}$-equivalence but not much.

Are there other possibilities?

Let Σ be a theory. We define an equivalence relation on κ^κ as follows: $\eta \cong_\Sigma \xi$ if $\mathcal{A}_\eta \cong \mathcal{A}_\xi$ or neither is a model of Σ, where $\mathcal{A}_\eta$ is the model coded by η.

(Friedman, H and Kulikov) $\cong_\Sigma$ is Borel iff Σ is classifiable and shallow.
If Σ is classifiable, then $\cong_\Sigma$ is Δ^1_1 (consistently if and only if).
If Σ is (e.g.) unstable, then $\cong_\Sigma$ is not Δ^1_1.
There is an unclassifiable Σ ($= \Sigma_\omega$) such that $\cong_\Sigma$ is Borel*.

Is it consistent that there is a unclassifiable theory with Δ^1_1 isomorphism relation?
Is it consistent that Borel*$=\Delta^1_1$?
Is it consistent that there is a theory whose isomorphism is not Borel*?

So just looking the topological complexity does not seem to improve classification (we will get more evidence on this later).

Borel* set are got by a straight forward generalization of Blackwell's game-theoretic definition of Borel sets.

There is also another way: Let $\mathcal{F}$ be the set of all functions from $\kappa^{<\kappa}$ to open sets. For $F \in \mathcal{F}$ and $\xi \in \kappa^\kappa$, $G^*(\xi, F)$ is a game length κ. At each round $i < \kappa$, first I does nothing and then II picks $\alpha_i \in \kappa$. II wins if for all $i < \kappa$, $\xi \in F((\alpha_j)_{j<i})$. Then $A \subseteq \kappa^\kappa$ is Σ^1_1 iff there is $F \in \mathcal{F}$ such that A is the set of all $\xi \in \kappa^\kappa$ such that $II \uparrow G^*(\xi, F)$.

$G^*(\xi, F)$ is a closed game.

Now $A \subseteq \kappa^\kappa$ is Borel* if there are $F \in \mathcal{F}$ and a κ^+, κ-tree T such that A is the set of all $\xi \in \kappa^\kappa$ such that $II \uparrow G^*_T(\xi, F)$.

Notice: II$\uparrow$ $G^*(\xi, F)$ iff II$\uparrow$ $G^*_T(\xi, F)$ for all κ^+, κ-trees T.

Recall: If $\kappa = \omega$, $\Delta^1_1 = Borel^* \subsetneq \Sigma^1_1$.

(Mekler and Väänänen) $\Delta^1_1 \subseteq Borel^* \subseteq \Sigma^1_1$.

(H and Kulikov) It is consistent that $\Delta^1_1 \subsetneq Borel^* \subsetneq \Sigma^1_1$.

(Friedman, H and Kulikov) In L, $Borel^* = \Sigma^1_1$.
In fact, every Σ^1_1-set is Wadge-reducible to the set of all $\eta \in \kappa^\kappa$ such that $\eta^{-1}(0) \cap S^\kappa_\omega$ is non-stationary.

Back to the classification.

Let $\mathcal{E}$ be the set of all Σ^1_1-equivalence relations on κ^κ. Idea (Friedman, H and Kulikov): Classify theories Σ by the position of their isomorphism relation $\cong_\Sigma$ in the partial order $(\mathcal{E}, \leq_\mathcal{B})$.

Notice that this does not work for $\kappa = \omega$: (Koerwien) There is an ω-stable classifiable depth 2 theory Σ_K such that in countable models $\cong_{\Sigma_K}$ is not Borel. On the other hand, e.g. $\cong_{DLO}$ is very simple and $\cong_{DLO} \leq_B \cong_{\Sigma_K}$ (by ω-categoricity).

Are classifiable theories below unclassifiable?

For $2 \leq \lambda \leq \kappa$ and regular $\mu < \kappa$, E^{λ}_{μ} is an equivalence relation on λ^{κ} such that $\eta E^{\lambda}_{\mu} \xi$ if the set $\{\alpha \in S^{\kappa}_{\mu} | \ \eta(\alpha) \neq \xi(\alpha)\}$ is non-stationary.

(Friedman, H and Kulikov)(Here one needs to assume e.g. that κ is a successor of a regular cardinal.)
If Σ is not classifiable, then there is regular $\mu < \kappa$ such that $E^2_{\mu} \leq_B \cong_{\Sigma}$. Consistently if and only if.

Does ZFC prove this if and only if?

(H, Kulikov and Moreno) If Σ is classifiable, then for all regular $\mu < \kappa$, $\cong_{\Sigma} \leq_B E^{\kappa}_{\mu}$. If in addition $\diamondsuit(S^{\kappa}_{\mu})$ holds, then E^{κ}_{μ} can be replaced by E^2_{μ}.

Thus under suitable diamonds (and...), if Σ is classifiable and Σ' is not, then $\cong_{\Sigma} \leq_B \cong_{\Sigma'}$. In fact, this gap between classifiable and unclassifiable can be forced to be big.

Does ZFC alone prove this?

Are there differences between unclassifiable theories?

(H, Kulikov and Moreno) In L, E^2_μ is Σ^1_1-complete for all regular $\mu < \kappa$.

In L, if κ is a successor of a regular uncountable cardinal, then $\cong_\Sigma$ is Σ^1_1-complete for all unclassifiable Σ.

Is it consistent that there is an unclassifiable Σ such that $\cong_\Sigma$ is not ISO-complete?
Clearly $\cong_\Sigma$ is ISO-complete e.g. if Σ is the theory of random graphs.

So at least consistently, the set of unclassifiable theories appears unclassifiable. Is this really the case?

I believe that most model theorists think that some unclassifiable theories are more unclassifiable than others.

Perhaps we do not see anything, because we look at cases in which our universe allows 'unrealistic' codings? Perhaps we should look at generic universe and a generic κ? Say κ is inaccessible and the universe is rich i.e. far from L.

H and Moreno and Moreno have looked a bit the inaccessible case: There are natural properties, ocp ($\sim$ didip) and sdop, of unclassifiable theories that push the isomorphism above isomorphisms of classifiable theories.

(E.g. Σ_ω has ocp and differentially closed fields have sdop.)

What other ways are there to classify unclassifiables?