On preserving AD by forcing

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Joint work with Nam Trang

Advertisement: Workshop in Kyoto!

In fall 2019, we will organize a set theory workshop in Kyoto, Japan.
You are all welcome to attend it!

From now on...

We work in ZF+$\mathsf{DC}_{\mathbb{R}}$.

The choice priciple $\mathsf{DC}_{\mathbb{R}}$ states the following:

For any $A \subseteq \mathbb{R} \times \mathbb{R}$, if $(\forall x \in \mathbb{R})\ (\exists x \in \mathbb{R})\ (x, y) \in A$,

$$\text{then } (\exists f : \omega \to \mathbb{R})\ (\forall n \in \omega)\ \big(f(n), f(n+1)\big) \in A.$$

Main Open Question

Question

Assume AD. Can one find a poset P which adds a new real while preserving AD?

The Axiom of Determinacy (AD) states that one of the players has a winning strategy for any Gale-Stewart game with a payoff set as a subset of the Baire space ω^ω.

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Example

Cohen forcing destroys AD.

Reason: The reals in the ground model does NOT have the Baire property in any Cohen forcing extension.

Main Result

Theorem (I., Trang)

Assume AD $+ V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ + "Every set of reals is ∞-Borel".
If a poset P increases Θ, then P destroys AD.

Definition

- A set of reals is ∞-Borel if it is λ-Borel for some λ.
- The ordinal Θ is defined as follows:

$$\Theta = \sup\{\gamma \mid \gamma \text{ is a surjective image of } \mathbb{R}\}$$

Three remarks on the main result

Remark

1. The assumption "$V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$" is essential, i.e., one can find a counter example of the theorem without this assumption.
2. The assumption "Every set of reals is ∞-Borel" is non-trivial. In fact, under this assumption, AD implies that every set of reals is Ramsey. On the other hand, it is still open whether AD itself implies that every set of reals is Ramsey.
3. In ZFC, $\Theta = (2^{\aleph_0})^+$, but under AD, Θ is quite large; it is a limit of measurable cardinals.

Background

Kunen: There is NO non-trivial & elementary $j\colon V \to V$ such that $(V, \in, j) \models$ ZFC.

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Hamkins et.al.: There is NO non-trivial & elementary $j\colon V \to V[G]$ such that $(V[G], \in, j) \vDash$ ZFC, where $V[G]$ is a set generic extension of V.

Woodin???: It is consistent to have $j\colon V \to V[G]$ as above if one demands $(V[G], \in, j) \vDash$ ZF only.
But in Woodin's example, $j \restriction \mathrm{Ord} = \mathrm{id}$.

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Question

How about demanding $(V[G], \in, j) \models$ ZF+AD and $V[G]$ has more reals than V?

Background ctd.

Theorem (Woodin)

Assume AD. If M is an inner model of ZF and V has more reals than M, then ω_1^M must be countable.

Corollary

If $j\colon V \to V[G]$ is non-trivial & elementary, and $V[G]$ is a model of AD and has more reals than V, then $\mathrm{crit}(j) = \omega_1^V$.

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Coming back to the first question:

Question

Assume AD. Can one find a poset P which adds a new real while preserving AD?

The main result stated again

Theorem (I., Trang)

Assume AD + $V = \mathrm{L}\big(\mathcal{P}(\mathbb{R})\big)$ + "Every set of reals is ∞-Borel".
If a poset P increases Θ, then P destroys AD.

Sketch of proof of the main result

For the proof, we use the following fact:

Fact (Woodin)

Assume AD + $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ + "Every set of reals is ∞-Borel". Then $\mathrm{HOD} = \mathrm{L}[X]$ for some $X \subseteq \Theta$ and V is definable in a set generic extension of HOD.

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Let G be P-generic over V and assume $V[G]$ was a model of AD.

Then $\Theta^{V[G]} > \Theta^V$ and $\Theta^{V[G]}$ is a limit of measurables in $V[G]$.

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But V is definable in a set generic extension of HOD.

So $X^\# \in \mathrm{HOD} = \mathrm{L}[X]$, contradiction!

THE END.