Kurepa trees and spectra of $\mathcal{L}_{\omega_1,\omega}$ sentences

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Outline

- Consistency results involving Kurepa trees.
- Application: analyzing the spectrum of an $\mathcal{L}_{\omega_1,\omega}$ sentence.

Motivation

Let ϕ be an an $\mathcal{L}_{\omega_1,\omega}$ sentence. The **spectrum** of ϕ is the set of all cardinalities of models of ϕ i.e.

$$Spec(\phi) = \{\kappa \mid \exists M \models \phi, |M| = \kappa\}$$

If $Spec(\phi) = [\aleph_0, \kappa]$, then ϕ **characterizes** κ.
General question: which cardinals can be characterized?
Some facts:

- (Morley, Lopez-Escobar) Let Γ be a countable set of $\mathcal{L}_{\omega_1,\omega}$ sentences. If Γ has models of cardinality $\beth_\alpha$ for all $\alpha < \omega_1$, then it has models in all infinite cardinalities.
- (Hjorth, 2002) For all $\alpha < \omega_1$, $\aleph_\alpha$ is characterized by a complete $\mathcal{L}_{\omega_1,\omega}$ sentence.

Corollary: Under GCH, $\aleph_\alpha$ is characterized by a complete $\mathcal{L}_{\omega_1,\omega}$ sentence iff $\alpha < \omega_1$.

Motivation

Corollary

Under GCH, $\aleph_\alpha$ is characterized by a complete $\mathcal{L}_{\omega_1,\omega}$ sentence iff $\alpha < \omega_1$.

Question:
Can there exists an $\mathcal{L}_{\omega_1,\omega}$ sentence that characterizes $\aleph_{\omega_1}$? (Under failure of GCH)
Answer: yes.

A conjecture of Shelah's: If $\aleph_{\omega_1} < 2^{\aleph_0}$, then any $\mathcal{L}_{\omega_1,\omega}$ sentence which has models of size $\aleph_{\omega_1}$ also has models of size $2^{\aleph_0}$.

We show: $2^{\aleph_0}$ cannot be replaced by $2^{\aleph_1}$ in the above.

The model theoretic application

We show the following:
There exists an $\mathcal{L}_{\omega_1,\omega}$ sentence ϕ, for which it is consistent with ZFC that:

1. ϕ characterizes $\aleph_{\omega_1}$, i.e. it has spectrum $[\aleph_0, \aleph_{\omega_1}]$.
2. $2^{\aleph_0} < \aleph_{\omega_1} < 2^{\aleph_1}$ and ϕ has models of size $\aleph_{\omega_1}$, but not $2^{\aleph_1}$.
3. The spectrum of ϕ can be $[\aleph_0, 2^{\aleph_1})$ where $2^{\aleph_1}$ is weakly inaccessible.

Note: this is the first example where the spectrum of a sentence can be both right-open and right-closed.

We define ϕ to code a Kurepa tree.

Kurepa trees

Definition

T is a **Kurepa tree** if T has countable levels, height $\aleph_1$, and at least $\aleph_2$ many cofinal branches.

For $\lambda > \omega_1$, $KH(\aleph_1, \lambda)$ is the statement that there exists a Kurepa tree with λ many branches.

$$\mathcal{B} := \sup\{\lambda \mid KH(\aleph_1, \lambda) \text{ holds }\}$$

Note that $\aleph_2 \leq \mathcal{B} \leq 2^{\aleph_1}$

Similarly, for any regular κ, can define κ-Kurepa trees, $KH(\kappa, \lambda)$ and $\mathcal{B}(\kappa)$, where κ is the height of the tree in place of $\aleph_1$; $\kappa^+ \leq \mathcal{B}(\kappa) \leq 2^\kappa$.

Kurepa trees

Theorem

There is an $\mathcal{L}_{\omega_1,\omega}$ *sentence* ϕ*, such that* ϕ *has a model of size* λ *iff* $\lambda \leq 2^{\aleph_0}$ *or there is a Kurepa tree with* λ *many branches (i.e.* $KH(\omega_1, \lambda)$*).*

In other words,

- If there are no Kurepa trees, $Spec(\phi) = [\aleph_0, 2^{\aleph_0}]$;
- If $\mathcal{B}$ is a maximum, then ϕ characterizes $\max(2^{\aleph_0}, \mathcal{B})$.

Consistency results

$\mathcal{B} := \sup\{\lambda \mid KH(\omega_1, \lambda) \text{ holds } \}$

Theorem

It is consistent with ZFC, that:

1. $2^{\aleph_0} < \aleph_{\omega_1} = \mathcal{B} < 2^{\aleph_1}$ *and there exist a Kurepa tree with* $\aleph_{\omega_1}$ *many branches.*
2. $\aleph_{\omega_1} = \mathcal{B} < 2^{\aleph_0}$ *and there exist a Kurepa tree with* $\aleph_{\omega_1}$ *many branches.*

Note that in both cases $\mathcal{B}$ *is a maximum.*

The model theoretic application:

Corollary

There is a $\mathcal{L}_{\omega_1,\omega}$ *sentence* ϕ*, which consistently:*

- *characterizes* $2^{\aleph_0}$*,*
- *characterizes* $\aleph_{\omega_1}$ *and* $2^{\aleph_0} < \aleph_{\omega_1}$*.*

An overview of the proof

Theorem

It is consistent with ZFC, that:

1. $2^{\aleph_0} < \aleph_{\omega_1} = \mathcal{B} < 2^{\aleph_1}$ *and there exist a Kurepa tree with* $\aleph_{\omega_1}$ *many branches.*
2. $\aleph_{\omega_1} = \mathcal{B} < 2^{\aleph_0}$ *and there exist a Kurepa tree with* $\aleph_{\omega_1}$ *many branches.*

Let $V \models ZFC + GCH$.
The forcing posets:

- Let $\mathbb{P}$ be the standard σ-closed, $\aleph_2$-c.c. poset to add a Kurepa tree with $\aleph_{\omega_1}$ many branches.
- Let $\mathbb{C} := Add(\omega, \aleph_{\omega_1+1})$

Then, we claim that

1. $V[\mathbb{P}]$ gives part (1)
2. $V[\mathbb{P} \times \mathbb{C}]$ gives part (2).

An overview of the proof

Some key points in the proof of (2):

- $\mathbb{P}$ adds a Kurepa tree with $\aleph_{\omega_1}$-many branches, showing that $\mathcal{B} \geq \aleph_{\omega_1}$.
- For $\alpha < \omega_1$, let $\mathbb{P}_\alpha$ be the restriction of $\mathbb{P}$ that adds the first $\aleph_\alpha$ many branches to the generic tree.

$\mathcal{B} \leq \aleph_{\omega_1}$:

- Let T be a Kurepa tree in $V[\mathbb{P}][\mathbb{C}]$.
- Then $T \in V[\mathbb{P}][Add(\omega, \omega_1)]$, for an appropriately chosen generic $Add(\omega, \omega_1)$.
- Every cofinal branch of T is in $V[\mathbb{P}_\alpha][Add(\omega, \omega_1)]$, for some $\alpha < \omega_1$.
- In $V[\mathbb{P}_\alpha][Add(\omega, \omega_1)]$, $2^{\omega_1} < \aleph_{\omega_1}$.

Then, by cardinal arithmetic, T cannot have more that $\aleph_{\omega_1}$ many branches.

Corollary: The sentence ϕ can characterize $\aleph_{\omega_1}$.

Consistency results

In the above theorem, we force $\mathcal{B}$ to be a maximum. And in part (1), $Spec(\phi) = [\aleph_0, \aleph_{\omega_1}]$.

Question: Can we have $\mathcal{B}$ be a supremum, but not a maximum? More generally, can the spectrum of an $\mathcal{L}_{\omega_1,\omega}$ sentence consistently be both right-hand closed and open?

It turns out, yes.

From a Mahlo cardinals, we force $\mathcal{B} = 2^{\aleph_1}$ and no Kurepa trees with $2^{\aleph_1}$ many branches.

Consistency results

$\mathcal{B}$ can be a supremum, not a maximum:

Theorem

From a Mahlo cardinal, it is consistent that $2^{\aleph_0} < \mathcal{B} = 2^{\aleph_1}$, for every $\kappa < 2^{\aleph_1}$, there is a Kurepa tree with at least κ many branches, but there is no Kurepa tree with $2^{\aleph_1}$ many branches.

Key notions in the proof:

- The forcing axiom GMA;
- a maximality principle, SMP;
- their consequences on $\mathbf{\Sigma}^1_1$ subsets of $\omega_1^{\omega_1}$.

GMA

A forcing axiom, defined by Shelah.
Some definitions:

Let κ be regular; a poset is **stationary κ^+-linked** if for every sequence $\langle p_\gamma \mid \gamma < \kappa^+ \rangle$, there is a regressive $f : \kappa^+ \to \kappa^+$, s.t. for almost all $\gamma, \delta \in \kappa^+ \cap \mathrm{cof}(\kappa)$, $f(\gamma) = f(\delta)$ implies that p_γ, p_δ are compatible.

Set Γ_κ to be the collection of all κ-closed, stationary κ^+-linked, well met posets with greatest lower bounds.

Definition

GMA_κ states that every $\mathbb{P} \in \Gamma_\kappa$ for every collection of dense sets $\mathcal{D} \subset \mathbb{P}$ with $|\mathcal{D}| < 2^\kappa$, there exists a $\mathcal{D}$-generic filter for $\mathbb{P}$.

SMP

A maximality principle, that generalizes GMA.

Definition

For a regular κ, $SMP_n(\kappa)$ states that:

- $\kappa^{<\kappa} = \kappa$;
- for any Σ_n formula ϕ, with parameters in $H(2^\kappa)$ and any $\mathbb{P} \in \Gamma_\kappa$, if
  for all κ-closed, κ^+-c.c. $\mathbb{Q} \in V[\mathbb{P}]$, $V[\mathbb{P}][\mathbb{Q}] \models \phi$, then
  ϕ is true in V.

SMP_κ means $SMP_n(\kappa)$ for all n.

Fact (Philipp Lücke): If $\kappa^{<\kappa} = \kappa$ and there is a Mahlo $\theta > \kappa$, then one can force $SMP(\kappa)$.

Some implications

Proposition

(Lücke)

1. *If* $\tau < 2^\kappa \to \tau^{<\kappa} < 2^\kappa$*, then* $SMP_1(\kappa)$ *iff* GMA_κ *and* $\kappa^{<\kappa} = \kappa$*.*
2. $SMP_2(\kappa)$ *implies that* 2^κ *is weakly inaccessible, and for all* $\tau < 2^\kappa$, $\tau^{<\kappa} < 2^\kappa$.
3. $SMP_2(\kappa)$ *implies that every* $\boldsymbol{\Sigma}^1_1$ *subset of* κ^κ *of cardinality* 2^κ *contains a perfect set.*

Here:
$A \subset \kappa^\kappa$ contains a perfect set if there is a continuous injection $g : 2^\kappa \to \kappa^\kappa$ with $\mathrm{ran}(g) \subset A$.
$A \subset \kappa^\kappa$ is $\boldsymbol{\Sigma}^1_1$ iff $A = p[T]$ for some tree $T \subset \kappa^{<\kappa} \times \kappa^{<\kappa}$.

a proof of

$SMP_2(\kappa)$ implies that every $\mathbf{\Sigma^1_1}$ subset of κ^κ of cardinality 2^κ contains a perfect set.
proof:
Let T be a tree in $\kappa^{<\kappa} \times \kappa^{<\kappa}$, we look at $p[T]$.
Set $\nu := 2^\kappa$, and let $\dot{\mathbb{Q}}$ be an $Add(\kappa, \nu^+)$ name for a κ-closed, κ^+ c.c poset. Denote $W := V[Add(\kappa, \nu^+)][\mathbb{Q}]$.
Note that V and W have the same cardinals.
Two cases:

1. $(p[T])^V \subsetneq (p[T])^W$, or
2. $W \models |p[T]| < 2^\kappa$

Case (1): can construct an embedding $g : 2^{<\kappa} \to \kappa^{<\kappa} \times \kappa^{<\kappa}$, $\mathrm{ran}(g) \subset T$ that witnesses $p[T]$ contains a perfect set.

So, $\phi :=$ "$|p[T]| < 2^\kappa$ or there is such an embedding " holds in W.
By $SMP_2(\kappa)$, ϕ holds in V.

Theorem

From a Mahlo cardinal, it is consistent that $2^{\aleph_0} < \mathcal{B} = 2^{\aleph_1}$, for every $\kappa < 2^{\aleph_1}$, there is a Kurepa tree with at least κ many branches, but there is no Kurepa tree with $2^{\aleph_1}$ many branches.

Proof.

Let V be a model of $SMP_2(\omega_1)$ (can be forced from a Mahlo). By the above, in V we have:

- GMA_{ω_1};
- CH, 2^{ω_1} is weakly inaccessible.
- Every $\boldsymbol{\Sigma}^1_1$ subset of $\omega_1^{\omega_1}$ of cardinality 2^{ω_1} contains a perfect set.

□

a proof of the theorem

Kurepa trees with (at least) κ many branches for all $\kappa < 2^{\aleph_1}$:

1. Let $\mathbb{P}$ be the standard poset to add such a tree.
2. $\mathbb{P}$ satisfies the hypothesis of GMA_{ω_1};
3. We need only κ many dense sets to meet to get the tree with κ branches.

So by GMA_{ω_1}, there is a Kurepa tree with at least κ many branches.

a proof of the theorem

No Kurepa trees with $2^{\aleph_1}$ many branches:
Let T be a Kurepa tree. Look at the set of branches, $[T]$.
Claim: $[T]$ is a closed set that does not contain a perfect set.
Pf:

- Let $g : 2^{\omega_1} \to 2^{\omega_1}$ be a continuous injection with $\mathrm{ran}(g) \subset [T]$.
- Construct $\langle p_s \mid s \in 2^{<\omega} \rangle$ and $\langle \alpha_n \mid n < \omega \rangle$, s.t.
  - $s' \supset s \to p_{s'} <_T p_s$; $|s| = n \to \alpha_n = \mathrm{dom}(p_s)$,
  - for each s, $p_{s^\frown 0} \neq p_{s^\frown 1}$.

  by induction on $|s|$.
- Then for $\alpha := \sup_n \alpha_n$, the α-th level of T has 2^ω many nodes:
  for $\eta \in 2^\omega$, set $p_\eta = \cup p_{\eta \restriction n}$.
- Contradiction with T being Kurepa.

So $|[T]| < 2^{\omega_1}$, as desired.

Some remarks

1. The idea of using Kurepa trees to get counter examples to the perfect set property goes back to Mekler and Väänänen.
2. A slightly weaker large cardinals hypothesis than a Mahlo suffices.
3. Our results generalize to κ-Kurepa trees for $\kappa \geq \aleph_2$.

Corollaries

Thm: can force $\mathcal{B} = 2^{\aleph_1}$ is not a maximum.

Corollary

There is an $\mathcal{L}_{\omega_1,\omega}$ sentence ϕ, such that under some mild large cardinals, it is consistent that the spectrum of ϕ is $[\aleph_0, 2^{\aleph_1})$, $2^{\aleph_0} < 2^{\aleph_1}$ and $2^{\aleph_1}$ is weakly inaccessible.

Corollary

Can have $2^{\aleph_0} < \aleph_{\omega_1} < 2^{\aleph_1}$ and sentence with models in $\aleph_{\omega_1}$, but no models in $2^{\aleph_1}$.

Recall Shelah's conjecture: If $\aleph_{\omega_1} < 2^{\aleph_0}$, then any $\mathcal{L}_{\omega_1,\omega}$ sentence which has models of size $\aleph_{\omega_1}$ also has models of size $2^{\aleph_0}$.

Corollary

$2^{\aleph_0}$ cannot be replaced by $2^{\aleph_1}$ in the above.

Summary of the properties of ϕ

Using consistency results about Kurepa trees, we produce an $\mathcal{L}_{\omega_1,\omega}$ sentence ϕ, for which it it consistent that:

1. ϕ characterizes $2^{\aleph_0}$
   (take a model with no Kurepa trees or with $\mathcal{B} < 2^{\aleph_0}$),
2. ϕ characterizes $\aleph_{\omega_1}$ and $2^{\aleph_0} < \aleph_{\omega_1}$
   (take the model with $2^{\aleph_0} < \mathcal{B} = \aleph_{\omega_1} < 2^{\aleph_1}$).
3. $Spec(\phi) = [\aleph_0, 2^{\aleph_1})$ and $2^{\aleph_0} < 2^{\aleph_1}$ and the latter is weakly inaccessible
   (use the last theorem, with $\mathcal{B} = 2^{\aleph_1}$ not a maximum).

More on ϕ

We get similar corollaries regarding the **maximal model** spectrum of ϕ, $MM - Spec(\phi) := \{\kappa \mid \exists M \models \kappa, |M| = \kappa, M \text{ is maximal}\}$, and the **amalgamation spectrum** of ϕ, $AP - Spec(\phi)$:
It is consistent that:

1. $MM - Spec(\phi) = \{\aleph_1, 2^{\aleph_0}\}$, $AP - Spec(\phi) = [\aleph_1, 2^{\aleph_0}]$ (take a model with no Kurepa trees),
2. $2^{\aleph_0} < \aleph_{\omega_1}$ and $AP - Spec(\phi) = [\aleph_1, \aleph_{\omega_1}]$;
3. $MM - Spec(\phi)$ is a cofinal subset of $[\aleph_1, 2^{\aleph_1})$,
   $AP - Spec(\phi) = [\aleph_1, 2^{\aleph_1})$
   (use the last theorem, with $\mathcal{B} = 2^{\aleph_1}$ not a maximum).

Open questions

1. Shelah's conjecture: If $\aleph_{\omega_1} < 2^{\aleph_0}$, then any $\mathcal{L}_{\omega_1,\omega}$ sentence which has models of size $\aleph_{\omega_1}$ also has models of size $2^{\aleph_0}$.
2. Recall, model existence in $\aleph_1$ is absolute for $\mathcal{L}_{\omega_1,\omega}$ sentences. **Open**: what about $\aleph_1$-amalgamation for $\mathcal{L}_{\omega_1,\omega}$ sentences? (By Shoenfield $\aleph_0$-amalgamation is absolute)