The Higher Cichoń Diagram

Thomas Baumhauer

(joint work with Martin Goldstern and Saharon Shelah)

TU Wien

thomas.baumhauer@tuwien.ac.at

Amsterdam, 23.8.2018

The Classical Cichoń Diagram

$$
\begin{array}{ccccccccc}
 & & \mathrm{cov}(\mathbf{n}) & \longrightarrow & \mathrm{non}(\mathbf{m}) & \longrightarrow & \mathrm{cof}(\mathbf{m}) & \longrightarrow & \mathrm{cof}(\mathbf{n}) \longrightarrow \mathfrak{c} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & & & \mathfrak{b} & \longrightarrow & \mathfrak{d} & & \\
 & & & & \uparrow & & \uparrow & & \\
\omega_1 & \rightarrow & \mathrm{add}(\mathbf{n}) & \longrightarrow & \mathrm{add}(\mathbf{m}) & \longrightarrow & \mathrm{cov}(\mathbf{m}) & \longrightarrow & \mathrm{non}(\mathbf{n})
\end{array}
$$

$\mathbf{n} \ldots$ null sets, $\mathbf{m} \ldots$ meager sets.

Generalisation to inaccessible $\kappa > \omega$

Straightforward generalisation:

- "$X \subseteq 2^\kappa$ is meager (in $<\kappa$-box product topology)".
- "$X \subseteq \kappa^\kappa$ is an unbounded/dominating family".

What about null sets?

Generalized Random Forcing $\mathbb{Q}_\kappa$ – Shelah (2012) [Sh:1004]

$p \subseteq 2^{<\kappa}$ is a condition of $\mathbb{Q}_\kappa$ iff there exist $\tau \in 2^{<\kappa}$, $S \subseteq \kappa$, $\langle N_\delta : \delta \in S \rangle$ such that:

- p is a tree with stem τ.
- Every node $\eta \trianglerighteq \tau$ has both successors in p.
- For limit ordinals $\delta \notin S$ no branches die out at level δ.
- S is a set of inaccessible cardinals such that $S \cap \alpha$ is not stationary in α for any $\alpha \leq \kappa$, $\mathrm{cf}(\alpha) > \omega$ ("nowhere stationary").
- For $\delta \in S$: $N_\delta \subseteq 2^\delta$ is a δ-null set. Definition: later.
- For $\delta \in S$: branches that lie in N_δ die out, all others continue.

The Generalized Random Forcing: Properties

$\mathbb{Q}_\kappa$ is κ-linked, hence: κ^+**-c.c**.

$\mathbb{Q}_\kappa$ is (strategically) κ**-complete**.

If κ is weakly compact then $\mathbb{Q}_\kappa$ is κ^κ**-bounding**.

Compare: random forcing is c.c.c, σ-complete, ω^ω-bounding.

The Generalized Null Ideal

Let $\mathcal{I} \subseteq \mathbb{Q}_\kappa$ be predense. We define:

$$\mathsf{set}_1(\mathcal{I}) = \bigcup_{p \in \mathcal{I}} [p] \qquad \mathsf{set}_0(\mathcal{I}) = 2^\kappa \setminus \mathsf{set}_1(\mathcal{I})$$

$\mathbf{wnull}_\kappa = \langle \{ \mathsf{set}_0(\mathcal{I}) : \mathcal{I} \subseteq \mathbb{Q}_\kappa \text{ is predense} \} \rangle$.

Define $\mathbf{null}_\kappa$ to be the $<\kappa^+$ closure of $\mathbf{wnull}_\kappa$.

The Generalized Null Ideal (Cont)

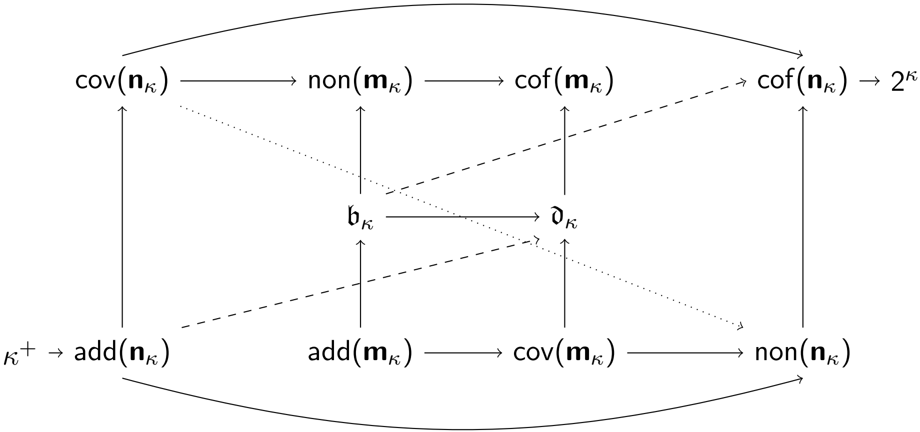
Let $\dot{\eta}$ be the name for the canonical κ-real added by $\mathbb{Q}_\kappa$ ("random real").

Theorem

Let $\mathbf{B} \subseteq 2^\kappa$ *be* κ*-Borel. Then:*

$$\mathbf{B} \in \mathbf{null}_\kappa \quad \Leftrightarrow \quad \Vdash_{\mathbb{Q}_\kappa} \dot{\eta} \notin \mathbf{B}.$$

The Higher Cichoń Diagram



$\mathbf{n}_\kappa \ldots$ κ-null sets, $\mathbf{m}_\kappa \ldots$ κ-meager sets.

(Dotted/dashed arrows for visual clarity.)

Consistency Results

Done: all vertical separations. (for $2^\kappa = \kappa^{++}$)

CON($\text{add}(\mathbf{null}_\kappa) > \mathfrak{b}_\kappa$)? (for sufficiently strong κ, work in progress)

Comparision to Classical Case

- Generalisation: κ-centered$_{<\kappa}$ forcings do not add κ-random reals (e.g. Hechler forcing).
- New concepts: the ideal of nowhere stationary sets appears.
- Violation of Fubini property:
    - Anti-symmetry: If η_1 is random over $\mathbf{V}$ and η_2 is random over $\mathbf{V}[\eta_1]$ then η_1 is not not random over $\mathbf{V}[\eta_2]$.
    - New inequalities: $\mathrm{cov}(\mathbf{null}_\kappa) \leq \mathrm{non}(\mathbf{null}_\kappa)$.
- Missing inequalities: $\mathrm{CON}(\mathrm{add}(\mathbf{null}_\kappa) > \mathfrak{b}_\kappa)$?
- New problems: preserve weak compactness, indestructibility.