More ZFC inequalities between cardinal invariants

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Outline

1. Eventual difference and $\mathfrak{a}_e(\kappa)$, $\mathfrak{a}_p(\kappa)$, $\mathfrak{a}_g(\kappa)$;
2. Generalized Unsplitting and Domination;
3. On some (consistency) results regarding generalized (un-)boundedness and splitting;

Eventual Difference

Almost disjointness

$\mathfrak{a}(\kappa)$ is the min size of a max almost disjoint $\mathscr{A} \subseteq [\kappa]^\kappa$ of size $\geq \kappa$

Relatives

- $\mathfrak{a}_e(\kappa)$ is the min size of max, eventually different family $\mathscr{F} \subseteq {}^\kappa\kappa$,
- $\mathfrak{a}_p(\kappa)$ is the min size of a max, eventually different family $\mathscr{F} \subseteq S(\kappa) := \{f \in {}^\kappa\kappa : f \text{ is bijective}\}$,
- $\mathfrak{a}_g(\kappa)$ is the min size of a max, almost disjoint subgroup of $S(\kappa)$.

Roitman Problem

Is it consistent that $\mathfrak{d} < \mathfrak{a}$?

- Yes, if $\aleph_1 < \mathfrak{d}$ (Shelah's template construction).
- Open, if $\aleph_1 = \mathfrak{d}$.

Is it consistent that $\mathfrak{d} = \aleph_1 < \mathfrak{a}_g$?

Roitman for Cofinitary Groups

Hrusak, Steprans, Zhang

Given a cofinitary group $\mathscr{G}$ of cardinality $\leq \omega_1$, there is an ${}^\omega\omega$-bounding proper poset which adjoins a generic permutation g such that $\langle \mathscr{G} \cup \{g\} \rangle$ is cofinitary. Additionally the poset has weak diagonalization. Thus consistently $\mathfrak{d} = \aleph_1 < \mathfrak{a}_g = \aleph_2$.

One of the major differences between $\mathfrak{a}$ and its relatives, is their relation to $\mathfrak{non}(\mathscr{M})$.

- While $\mathfrak{a}$ and $\mathfrak{non}(\mathscr{M})$ are independent, we have
- $\mathfrak{non}(\mathscr{M}) \leq \mathfrak{a}_e, \mathfrak{a}_p, \mathfrak{a}_g$ (Brendle, Spinas, Zhang),

Theorem

Let κ be regular uncountable. Then

- (Blass, Hyttinen, Zhang) $\mathfrak{b}(\kappa) \leq \mathfrak{a}(\kappa), \mathfrak{a}_e(\kappa), \mathfrak{a}_p(\kappa), \mathfrak{a}_g(\kappa)$;
- (Hyttinen) Let $\mathfrak{nm}(\kappa)$ be the least size of a family $\mathscr{F} \subseteq {}^\kappa\kappa$ such that $\forall g \in {}^\kappa\kappa \exists f \in \mathscr{F}$ with $|\{\alpha \in \kappa : f(\alpha) = g(\alpha)\}| = \kappa$. Whenever κ is a successor, we have $\mathfrak{mn}(\kappa) = \mathfrak{b}(\kappa)$.

What we still do not know...

Even though Con($\mathfrak{a} < \mathfrak{a}_g$), both

- the consistency of $\mathfrak{a}_g < \mathfrak{a}$, as well as
- the inequality $\mathfrak{a} \leq \mathfrak{a}_g$ (in ZFC)

are open.

Roitman in the Uncountable

Theorem (Blass, Hyttinen and Zhang)

Let $\kappa \geq \aleph_1$ be regular uncountable. Then

$$\mathfrak{d}(\kappa) = \kappa^+ \Rightarrow \mathfrak{a}(\kappa) = \kappa^+.$$

The cofinitary groups analogue

Clearly, the result does not have a cofinitary group analogue for $\kappa = \aleph_0$, since $\mathfrak{d} = \aleph_1 < \mathfrak{a}_g = \mathfrak{a}_g(\aleph_0) = \aleph_2$ is consistent. Nevertheless the question remains of interest for uncountable κ: Is it consistent that

$$\mathfrak{d}(\kappa) = \kappa^+ \Rightarrow \mathfrak{a}_g(\kappa) = \kappa^+?$$

Club unboundedness

Theorem (Raghavan, Shelah)

Let κ be regular uncountable. Then $\mathfrak{b}(\kappa) = \kappa^+ \Rightarrow \mathfrak{a}(\kappa) = \kappa^+$.

Club unboundedness

1. Let κ be regular uncountable. For $f, g \in {}^\kappa\kappa$ we say that $f \leq_{cl} g$ iff $\{\alpha < \kappa : g(\alpha) < f(\alpha)\}$ is non-stationary.
2. $\mathscr{F} \subseteq {}^\kappa\kappa$ is $\leq_{cl}$-unbounded if $\neg\exists g \in {}^\kappa\kappa \forall f \in \mathscr{F}(f \leq_{cl} g)$.
3. $\mathfrak{b}_{cl}(\kappa) = \min\{|F| : F \subseteq {}^\kappa\kappa$ and F is cl-unbounded$\}$

Theorem (Cummings, Shelah)

If κ is regular uncountable then $\mathfrak{b}(\kappa) = \mathfrak{b}_{\mathrm{cl}}(\kappa)$.

Higher eventually different analogues

Theorem(F., D. Soukup, 2018)

Suppose $\kappa = \lambda^+$ for some infinite λ and $\mathfrak{b}(\kappa) = \kappa^+$. Then $\mathfrak{a}_e(\kappa) = \mathfrak{a}_p(\kappa) = \kappa^+$. If in addition $2^{<\lambda} = \lambda$, then $\mathfrak{a}_g(\kappa) = \kappa^+$.

Remark

The case of $\mathfrak{a}_e(\kappa)$ has been considered earlier. The above is a strengthening of each of the following:

- $\mathfrak{d}(\kappa) = \kappa^+ \Rightarrow \mathfrak{a}_e(\kappa) = \kappa^+$ for κ successor (Blass, Hyttinen, Zhang)
- $\mathfrak{b}(\kappa) = \kappa^+ \Rightarrow \mathfrak{a}_e(\kappa) = \kappa^+$ proved by Hyttinen under additional assumptions.

Outline: $\mathfrak{b}(\kappa)=\kappa^+ \Rightarrow \mathfrak{a}_e(\kappa)=\kappa^+$

- Let $\{f_\delta : \delta<\kappa^+\}$ witness $\mathfrak{b}_{\mathrm{cl}}(\kappa)=\kappa^+$.
- Fix bijections $e_\delta:\kappa\to\delta$ for $\kappa\le\delta<\kappa^+$ and $d_\alpha:\alpha\to\lambda$ for $\lambda\le\alpha<\kappa$.
- Define $h_{\delta,\zeta}\in{}^\kappa\kappa$ for $\delta<\kappa^+$, $\zeta<\lambda$ by induction on $\delta<\kappa^+$:
- For $\mu<\kappa$, let $\mathbb{H}_\delta(\mu)=\{h_{\delta',\zeta'}:\delta'\in\mathrm{ran}(e_\delta\restriction\mu),\zeta'\in\lambda\}$. Then $\mathbb{H}_\delta(\mu)$ and so $H_\delta(\mu)=\{h(\mu):h\in\mathbb{H}_\delta(\mu)\}$ are of size $<\kappa$ and

  $$f^*_\delta(\mu)=\max\{f_\delta(\mu),\min\{\alpha:|\alpha\backslash H_\delta(\mu)|=\lambda\}\}$$

  is well-defined. Now, for each $\zeta<\lambda$, define $h_{\delta,\zeta}(\mu):=\beta$ where β is such that

  $$d_{f^*_\delta(\mu)}[\beta\cap(f^*_\delta(\mu)\backslash H_\delta(\mu))]$$

  is of order type ζ.

Claim: $\{h_{\delta,\zeta}\}_{\delta<\kappa^+,\zeta<\lambda}$ is κ-med.

1. (κ-ed) Fix δ. If $\zeta < \zeta'$ then by definition for all $\mu < \kappa$, $h_{\delta,\zeta}(\mu) \neq h_{\delta,\zeta'}(\mu)$. Suppose $\delta' < \delta$ and $\zeta, \zeta' < \lambda$. If

$$\delta' \in \mathrm{ran}(e_\delta \restriction \mu),$$

then $h_{\delta',\zeta'} \in \mathbb{H}_\delta(\mu)$ and so $h_{\delta',\zeta'}(\mu) \neq h_{\delta,\zeta}(\mu)$. Because e_δ is a bijection, there is μ' such that $\delta' \in \mathrm{ran}(e_\delta \restriction \mu')$.

2. (Maximality) Let $h \in {}^\kappa\kappa$ and $\delta < \kappa^+$ s.t. $S = \{\mu : h(\mu) < f_\delta(\mu)\}$ is stationary. There is stationary $S_0 \subseteq S$ such that

$$\Big(h(\mu) \in H_\delta(\mu) \text{ for all } \mu \in S_0\Big) \text{ or } \Big(h(\mu) \notin H_\delta(\mu) \text{ for all } \mu \in S_0\Big).$$

In either case, there are δ, ζ such that $h_{\delta,\zeta}(\mu) = h(\mu)$ for stationarily many $\mu \in S_0$. □

Questions

- Is it true that $\mathfrak{b}(\kappa) = \kappa^+$ implies that $\mathfrak{a}_e(\kappa) = \mathfrak{a}_p(\kappa) = \kappa^+$ if κ is not a successor?
- Can we drop the requirement $2^{<\lambda} = \lambda$ from the proof of $\mathfrak{b}(\kappa) = \kappa^+ \Rightarrow \mathfrak{a}_g(\kappa) = \kappa^+$?

Definition

Let κ be regular uncountable.

- A family $F \subseteq [\kappa]^\kappa$ is splitting if for every $B \in [\kappa]^\kappa$ there is $A \in F$ such that $|B \cap A| = |B \cap (\kappa \setminus A)| = \kappa$, i.e. A splits B. Then

$$\mathfrak{s}(\kappa) := \min\{|F| : F \text{ is splitting}\}.$$

- A family $F \subseteq [\kappa]^\kappa$ is unsplit if there is no $B \in [\kappa]^\kappa$ which splits every element of F. Then

$$\mathfrak{r}(\kappa) := \min\{|F| : F \text{ is unsplit }\}.$$

Theorem (Raghavan, Shelah)

Let κ be regular uncountable. Then $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$.

Thus splitting and unboundedness at κ behave very differently than splitting and unboundedness at ω, as it is well known that $\mathfrak{s}$ and $\mathfrak{b}$ are independent. However, of interest becomes the following question: Does the above inequality dualize? Is it true that for every regular uncountable κ, $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$?

Theorem (Raghavan, Shelah)

Let $\kappa \geq \beth_\omega$ be regular. Then $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$.

Club domination

1. $\mathscr{F} \subseteq {}^{\kappa}\kappa$ is $\leq_{cl}$-dominating if $\forall g \in {}^{\kappa}\kappa \exists f \in \mathscr{F}(g \leq_{cl} f)$.
2. $\mathfrak{d}_{\text{cl}}(\kappa) = \min\{|F| : F \subseteq {}^{\kappa}\kappa \wedge F \text{is cl-dominating}\}$.

Almost always the same

Theorem (Cummings, Shelah)

$\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathrm{cl}}(\kappa)$ whenever $\kappa \geq \beth_\omega$ regular.

Outline: $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$ for $\kappa \geq \beth_\omega$ regular

- For κ be regular uncountable and $A \in [\kappa]^\kappa$, let $e_A : \kappa \to A$ be the order isomorphism from $\langle \kappa, \in \rangle$ to $\langle A, \in \rangle$, and $s_A : \kappa \to A$ be defined as follows: $s_A(\alpha) = \min(A \backslash (\alpha + 1))$
- Take unsplit $F \subseteq [\kappa]^\kappa$ of size $\mathfrak{r}(\kappa)$.
- If $\exists$ club E_1 such that $\forall$ club $E_2 \subseteq E_1$ there is $A \in F$ with $A \subseteq^* \bigcup_{\xi \in E_2} [\xi, s_{E_1}(\xi))$ then $\{s_A \circ s_{E_1} : A \in F\}$ is $\leq^*$-dominating.
- Otherwise, F has the *RS*-property: For every club E_1, there is a club $E_2 \subseteq E_1$ such that for all $A \in F$, $A \not\subseteq^* \bigcup_{\xi \in E_2} [\xi, s_{E_1}(\xi))$.
- We will show that $\{s_A : A \in F\}$ is $\leq_{cl}$-dominating.

Outline: $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$ for $\kappa \geq \beth_\omega$ regular

- Take $f \in {}^\kappa\kappa$ and let E_1 be an f-closed club. Pick E_2-given by RS.
- If for all $A \in F$, $|A \cap \bigcup_{\xi \in E_2} [\xi, s_{E_1}(\xi))| = \kappa$, then $\bigcup_{\xi \in E_2} [\xi, s_{E_1}(\xi))$ splits F, contradicting F is unsplit.
- Thus there are $A \in F$, $\delta < \kappa$ with $A \backslash \delta \subseteq \kappa \backslash \bigcup_{\xi \in E_2} [\xi, s_{E_1}(\xi))$.
- Take any $\xi \in E_2 \backslash \delta$. Then, since $s_A(\xi) \in A$ and $A \cap [\xi, s_{E_1}(\xi)) = \emptyset$, we get $f(\xi) < s_{E_1}(\xi) \leq s_A(\xi)$.
- Therefore $\{s_A : A \in F\}$ is $\leq_{cl}$-dominating, and so $\mathfrak{d}_{cl}(\kappa) \leq |F| = \mathfrak{r}(\kappa)$.
- Since $\kappa \geq \beth_\omega$, $\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathrm{cl}}(\kappa)$ and so $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$. □

Strong Unsplitting: $\mathfrak{r}_\sigma(\kappa)$

Definition

Recall that $\mathfrak{r}_\sigma(\kappa)$ is the least size of a $F \subseteq [\kappa]^\kappa$ such that there is no countable $\{B_n : b \in \omega\}$ such that every $A \in F$ is split by some B_n.

Remark

If $\mathfrak{r}_\sigma(\kappa)$ exists, then $\mathfrak{r}(\kappa) \leq \mathfrak{r}_\sigma(\kappa)$.

Strong Unsplitting: $\mathfrak{r}_\sigma(\kappa)$

Theorem (Zapletal)

If $\aleph_0 < \kappa \leq 2^{\aleph_0}$ then there is a countable $\mathscr{B}$ splitting all $A \in [\kappa]^\kappa$.

Proof:

Let $f : \kappa \to 2^\omega$ be an injection and for each $s \in 2^{<\omega}$ let $B_s := \{\alpha < \kappa : s \subseteq f(\alpha)\}$. Then $\{B_s : s \in 2^{<\omega}\}$ splits all $A \in [\kappa]^\kappa$. Indeed. Suppose $A \in [\kappa]^\kappa$ is not split by any B_s. Then $S = \{s \in {}^{<\omega}2 : |A \cap B_s| = \kappa\}$ does not contain incompatible elements. However, then there is at most one α such that if $s \in S$ then $s \subseteq f(\alpha)$, and so $A \subseteq \{\alpha\} \cup \bigcup_{s \in 2^{<\omega} \setminus S} A \cap B_s$. Therefore $|A| < \kappa$, which is a contradiction. □

$\mathfrak{d}(\kappa) \leq \mathfrak{r}_\sigma(\kappa)$

Remark

Thus $\mathfrak{r}_\sigma(\kappa)$ does not exist for $\aleph_0 < \kappa \leq 2^{\aleph_0}$. However:

Theorem(F., D. Soukup, 2018)

If $\kappa > 2^{\aleph_0}$ is regular, then $\mathfrak{r}_\sigma(\kappa)$-exists and $\mathfrak{d}(\kappa) \leq \mathfrak{r}_\sigma(\kappa)$.

Characterizing $\mathfrak{d}(\kappa)$

Among others, the above result leads to a new characterization of $\mathfrak{d}(\kappa)$ for regular uncountable κ.

Finitely Reaping Number

Definition

Let $\mathfrak{fr}$ denote the minimal size of a family $\mathscr{I}$ of partitions of ω into finite sets, so that there is no single $A \in [\omega]^\omega$ with the property that for each partition $\{I_n\}_{n\in\omega} \in \mathscr{I}$ both

$$\{n \in \omega : I_n \subseteq A\} \text{ and } \{n \in \omega : I_n \cap A = \emptyset\}$$

are infinite.

Theorem (Brendle)

$\mathfrak{fr} = \min\{\mathfrak{d}, \mathfrak{r}\}$.

Generalizations: $\mathfrak{fr}(\kappa)$

Definition

For κ regular uncountable, let $\mathfrak{fr}(\kappa)$ denote the minimal size of a family $\mathscr{E}$ of clubs, so that there is no $A \subseteq \kappa$ such that for all $E \in \mathscr{E}$ both

$$\{\xi \in E : [\xi, s_E(\xi)) \subseteq A\} \text{ and } \{\xi \in E : [\xi, s_E(\xi)) \cap A = \emptyset\}$$

have size κ. We say that A interval-splits E.

Characterization of $\mathfrak{d}(\kappa)$

Theorem (F., D. Soukup, 2018)

Let κ be a regular uncountable. Then $\mathfrak{d}(\kappa)$ is the minimal size of a family $\mathscr{E}$ of clubs so that there is no countable $\mathscr{A} \subseteq [\kappa]^\kappa$ with the property that for each $E \in \mathscr{E}$ there is $A \in \mathscr{A}$ with the property that both

$$\{\xi \in E : [\xi, s_E(\xi)) \subseteq A\} \text{ and } \{\xi \in E : [\xi, s_E(\xi)) \cap A = \emptyset\}$$

have size κ.

Remark

With other words, $\mathfrak{d}(\kappa) = \mathfrak{fr}_\sigma(\kappa)$ for κ regular uncountable.

Outline: $\mathfrak{fr}_\sigma(\kappa) \leq \mathfrak{d}(\kappa)$

- Let $\mathscr{F} \subseteq {}^\kappa\kappa$ be dominating, $|\mathscr{F}| = \mathfrak{d}(\kappa)$.
- For each $f \in \mathscr{F}$ fix an f-closed club E_f and let $\mathscr{E} := \{E_f : f \in \mathscr{F}\}$.
- Let $\mathscr{A} \subseteq [\kappa]^\kappa$ be countable and let $g = \sup\{s_A : A \in \mathscr{A}\}$. Find $f \in \mathscr{F}$ with $g \leq^* f$.
- Then for each $A \in \mathscr{A}$, the set $\{\xi \in \kappa : [\xi, s_{E_f}(\xi)) \cap A = \emptyset\}$ is bounded.
- Thus, there is no countable $\mathscr{A}$ such that each E_f is split by some $A \in \mathscr{A}$. Thus $\mathfrak{fr}_\sigma(\kappa) \leq |\mathscr{E}| = \mathfrak{d}(\kappa)$.

Outline: $\mathfrak{d}(\kappa) \leq \mathfrak{fr}_\sigma(\kappa)$

- Let $|\mathscr{E}| < \mathfrak{d}(\kappa)$ be a family of clubs.
- Take $f \in {}^\kappa\kappa$ such that for all $E \in \mathscr{E}$ the set $\{\alpha < \kappa : s_E \circ s_E(\alpha) < f(\alpha)\}$ is unbounded in κ.
- Let D be an f-closed club. Then for each $E \in \mathscr{E}$ the set $X_E := \{\zeta \in D : (\exists \xi \in E)([\xi, s_E(\xi)) \subseteq [\zeta, s_D(\zeta)))\}$ is unbounded.
- Since $|\mathscr{E}| < \mathfrak{d}(\kappa) \leq \mathfrak{r}_\sigma(\kappa)$, there is a countable $\{B_n\}_{n\in\omega} \subseteq [\kappa]^\kappa$ so that each X_E is split by some B_n.
- For each $n \in \omega$, let $A_n := \bigcup\{[\zeta, s_D(\zeta)) : \zeta \in D \cap B_n\}$.
- Then each $E \in \mathscr{E}$ is interval-split by some A_n. $\square$

On cofinalities

Remark

It is a long-standing open problem if $\mathfrak{r}$ can be of countable cofinality. However, if $\mathrm{cf}(\mathfrak{r}) = \omega$ then $\mathfrak{d} \leq \mathfrak{r}$.

Theorem (F., Soukup, 2018)

If κ is regular, uncountable and $\mathrm{cf}(\mathfrak{r}(\kappa)) \leq \kappa$ then $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$.

Questions

- (Cummings-Shelah) Does $\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathsf{cl}}(\kappa)$ for all regular uncountable κ?
- (Raghavan-Shelah) Does $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$ for all regular uncountable κ?

Generalization of S. Hechler's result

Lemma (Cummings, Shelah)

Let κ be a regular uncountable. Then

$$\kappa^+ \leq \mathrm{cf}(\mathfrak{b}(\kappa)) = \mathfrak{b}(\kappa) \leq \mathrm{cf}(\mathfrak{d}(\kappa)) \leq 2^\kappa.$$

Triple Realizations

Theorem (Cummings, Shelah)

Assume $\kappa^{<\kappa} = \kappa$, GCH above κ and (β, δ, μ) such that

$$\kappa^+ \leq \beta = \mathrm{cf}(\beta) \leq \mathrm{cf}(\delta) \leq \mu \text{ and } \kappa < \mathrm{cf}(\mu).$$

Then there is a cardinal preserving generic extension in which $\mathfrak{b}(\kappa) = \beta$, $\mathfrak{d}(\kappa) = \delta$ and $2^\kappa = \mu$.

An iteration along a non-linear, well-founded index set of the generalized Hechler poset for adjoining a dominating real.

Theorem (Raghavan, Shelah)

Let κ be regular uncountable. Then $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$.

Let κ be regular uncountable. Then

- (Zapletal) $\mathfrak{s}(\kappa) \geq \kappa$ iff κ is inaccessible, and
- (Suzuki) $\mathfrak{s}(\kappa) > \kappa$ iff κ is weakly compact.

Theorem (Ben-Neria, Gitik)

$\mathfrak{s}(\kappa) > \kappa^+$ is equiconsistent with the existence of a measurable cardinal μ with Mitchell order at least μ^{++}.

Quadruple Realizations

Observation(F., Bag, 2018)

Let κ be a supercompact. Then consistently

$$\mathfrak{s}(\kappa) = \kappa^+ < \mathfrak{b}(\kappa) = \mu_\kappa < \mathfrak{d}(\kappa) = \nu_\kappa < 2^\kappa = \zeta_\kappa$$

for any admissible triple $\mu_\kappa < \nu_\kappa < \zeta_\kappa$.

Laver preparation, followed by adjoining κ^+ many κ-Cohen reals, followed by Cummings-Shelah construction.

Quadruple Realizations

Work in Progress (F., Shelah, 2018)

The above result generalizes to

$$\kappa^+ < \mathfrak{s}(\kappa) = \xi_\kappa < \mathfrak{b}(\kappa) = \mu_\kappa < \mathfrak{d}(\kappa) = \nu_\kappa < 2^\kappa = \zeta_\kappa$$

for any admissible quadruple $\xi_\kappa < \mu_\kappa < \nu_\kappa < \zeta_\kappa$.

Questions

- (Ben-Neria, Gitik) Is it consistent that $\mathfrak{s}(\kappa)$ is singular for some uncountable regular κ?
- Does $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa)$ for all regular uncountable κ?