Open Colorings, Perfect Sets and Games on Generalized Baire Spaces

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August 24, 2018

Open Colorings on Generalized Baire Spaces

Open coloring axioms for subsets of the κ-Baire space

Let κ be an infinite cardinal such that $\kappa^{<\kappa} = \kappa$. Let $X \subseteq {}^{\kappa}\kappa$.

$\mathrm{OCA}_{\kappa}(X)$:

Suppose $[X]^2 = R_0 \cup R_1$ is an open partition

(i.e. $\{(x, y) : \{x, y\} \in R_0\}$ is an open subset of $X \times X$).

Then either X is a union of κ-many R_1-homogeneous sets, or there exists an R_0-homogeneous set of size κ^+.

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i.e., there is a continuous embedding $f : {}^{\kappa}2 \to X$ whose image is R_0-homogeneous.

$\mathrm{OCA}^*_\kappa(X)$ for definable subsets X of ${}^\kappa\kappa$

Theorem (Feng, 1993)

1. *$\mathrm{OCA}^*_\omega(X)$ holds for all $\mathbf{\Sigma}^1_1$ subsets of ${}^\omega\omega$.*
2. *In Solovay's model, $\mathrm{OCA}^*_\omega(X)$ holds for all $X \subseteq {}^\omega\omega$.*

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Theorem (Sz., 2017)

In $\mathrm{Col}(\kappa, {<}\lambda)$-generic extensions, where $\lambda > \kappa$ is inaccessible, $\mathrm{OCA}^*_\kappa(X)$ holds for all $\mathbf{\Sigma}^1_1$ subsets $X \subseteq {}^\kappa\kappa$.

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These results give the exact consistency strength of these statements.

A game for open colorings

Definition

Let $X \subseteq {}^{\kappa}\kappa$ and let $R_0 \subseteq [X]^2$. $G^*_\kappa(X, R_0)$ is the following game.

I		i_0		i_1	$\dots$		i_α	$\dots$
II	u^0_0, u^1_0		u^0_1, u^1_1		$\dots$	u^0_α, u^1_α		$\dots$

II plays $u^0_\alpha, u^1_\alpha \in {}^{<\kappa}\kappa$. Then **I** chooses, by playing $i_\alpha < 2$.

Rules: for all $\beta < \alpha$ and $i < 2$ we have $u^i_\alpha \supset u^{i_\beta}_\beta$ and $N_{u^i_\alpha} \cap X \neq \emptyset$ and

$$N_{u^0_\alpha} \times N_{u^1_\alpha} \subseteq R_0$$

Player **II** wins the round iff $\bigcup_{\alpha<\kappa} u^{i_\alpha}_\alpha \in X$.

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Remark: For the trivial coloring $R_0 = [X]^2$,

$\mathcal{G}^*_\kappa\big(X, [X]^2\big) = \mathcal{G}^*_\kappa(X)$ is the perfect set game.

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$\mathbf{II}$ plays $u^0_\alpha, u^1_\alpha \in {}^{<\kappa}\kappa$. Then $\mathbf{I}$ chooses, by playing $i_\alpha < 2$.
Rules: for all $\beta < \alpha$ and $i < 2$ we have $u^i_\alpha \supset u^{i_\beta}_\beta$ and $N_{u^i_\alpha} \cap X \neq \emptyset$ and

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Proposition (Sz., 2017)

$\mathrm{OCA}^*_\kappa(X)$ holds iff $G^*_\kappa(X, R_0)$ is determined for all open $R_0 \subseteq [X]^2$.

Questions

- Is it consistent that all $\mathbf{\Sigma}^1_1$ subsets have the κ-perfect set property but there is a closed $X \subseteq {}^\kappa\kappa$ such that $\mathrm{OCA}^*_\kappa(X)$ does not hold?

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  Is OCA_κ consistent?

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- Let OCA_κ say: "$\mathrm{OCA}_\kappa(X)$ holds for all $X \subseteq {}^\kappa\kappa$".

  Is OCA_κ consistent? If so, how does it influence the structure of the κ-Baire space?

Perfect Sets and Games

Väänänen's perfect set game

Definition (Väänänen, 1991)

Let $X \subseteq {}^{\kappa}\kappa$, let $x_0 \in X$ and let $\omega \leq \gamma \leq \kappa$. Then $\mathcal{V}_\gamma(X, x_0)$ is the following game.

$\mathbf{I}$		δ_1		$\ldots$		δ_α		$\ldots$	
$\mathbf{II}$	x_0		x_1		$\ldots$		x_α		$\ldots$

$\mathbf{I}$ plays $\delta_\alpha < \kappa$ such that $\delta_\alpha > \delta_\beta$ for all $\beta < \alpha$, and $\delta_\alpha = \sup_{\beta<\alpha} \delta_\beta$ at limits α.

$\mathbf{II}$ responds with $x_\alpha \in X$ such that $x_\alpha \restriction \delta_{\beta+1} = \delta_\beta \restriction \delta_{\beta+1}$ but $x_\alpha \neq x_\beta$ for all $\beta < \alpha$.

Player $\mathbf{II}$ wins, if she can make all her γ moves.

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X is a γ-perfect set if **II** wins $\mathcal{V}_\gamma(X, x_0)$ for all $x_0 \in X$.

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Definition (Väänänen, 1991)

X is a γ-perfect set if $\mathbf{II}$ wins $\mathcal{V}_\gamma(X, x_0)$ for all $x_0 \in X$.

X is a γ-scattered set if $\mathbf{I}$ wins $\mathcal{V}_\gamma(X, x_0)$ for all $x_0 \in X$.

Väänänen's generalized Cantor-Bendixson theorem

(1) *Väänänen's generalized Cantor-Bendixson theorem*:

every closed subset of ${}^{\kappa}\kappa$ *is the (disjoint) union of a* κ*-perfect set and a* κ*-scattered set, which is of size* $\leq \kappa$.

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Proposition (Sz.)

The following statements are equivalent:

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- Väänänen (1991) showed that (1) is consistent relative to the existence of a measurable $\lambda > \kappa$.
- Galgon (2016) showed that (1) holds after Lévy-collapsing an inaccessible $\lambda > \kappa$ to κ^+.

A different definition of κ-perfectness

Definition

A subtree T of ${}^{<\kappa}\kappa$ is a strongly κ-perfect tree if T is $<\kappa$-closed and every node of T extends to a splitting node.

A set $X \subseteq {}^{\kappa}\kappa$ is a strongly κ-perfect set if $X = [T]$ for a strongly κ-perfect tree T.

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Proposition

Let X be a closed set of ${}^{\kappa}\kappa$.

$$X \text{ is } \kappa\text{-perfect} \iff X = \bigcup_{i \in I} X_i \text{ for strongly } \kappa\text{-perfect sets } X_i.$$

γ-perfect trees when $\omega \leq \gamma \leq \kappa$

γ-perfect trees and γ-scattered trees can be defined using a strong cut-and-choose game $\mathcal{G}_\gamma(T, t_0)$ played on subtrees T of ${}^{<\kappa}\kappa$ (Galgon, 2016).

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Remark: $\mathcal{G}_\gamma(T, t_0)$ is easier for player $\mathbf{I}$ and harder for player $\mathbf{II}$ to win than the perfect set game $\mathcal{G}^*_\gamma([T] \cap N_{t_0})$.

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In the $\gamma = \kappa$ case, $\mathcal{G}_\kappa(T, t_0)$ is equivalent to the $\mathcal{G}^*_\kappa([T] \cap N_{t_0})$.

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$$T \text{ is a } \kappa\text{-perfect tree} \iff [T] \text{ is a } \kappa\text{-perfect set.}$$

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If the κ-perfect set property holds, then

$$T \text{ is a } \kappa\text{-scattered tree} \iff [T] \text{ is a } \kappa\text{-scattered set.}$$

γ-perfect sets and trees when $\gamma < \kappa$

Theorem (Sz.)

Let T be a subtree of ${}^{<\kappa}\kappa$ and let $\omega \leq \gamma < \kappa$.

1. *If $[T]$ is a γ-perfect set, then T is a γ-perfect tree.*

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3. *If κ is weakly compact and $T \subseteq {}^{<\kappa}2$, then*

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   More generally: holds if κ has the tree property and T is a κ-tree.

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Question

Is it consistent that 3 holds for "scattered" instead of "perfect"?

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Analogues of 1–3 hold for "generalized Cantor-Bendixson ranks" for subsets of ${}^{\kappa}\kappa$ and for subtrees of ${}^{<\kappa}\kappa$ (see next 2 slides).

Generalizing Cantor-Bendixson ranks

Definition (Väänänen, 1991)

Let $X \subseteq {}^{\kappa}\kappa$, let $x_0 \in X$, and let U be a tree without κ-branches.
$\mathcal{V}_U(X, x_0)$ is the following game.

I		t_1, δ_1		$\ldots$		t_α, δ_α	$\ldots$
II	x_0		x_1		$\ldots$	x_α	$\ldots$

I plays $t_\alpha \in U$ such that $t_\alpha >_U t_\beta$ and $\delta_\alpha < \kappa$ such that $\delta_\alpha > \delta_\beta$ for all $\beta < \alpha$, and $\delta_\alpha = \sup_{\beta<\alpha} \delta_\beta$ at limits α.

II responds with $x_\alpha \in X$ such that $x_\alpha \restriction \delta_{\beta+1} = x_\beta \restriction \delta_{\beta+1}$ but $x_\alpha \neq x_\beta$ for all $\beta < \alpha$.

The first player who can not move loses, and the other player wins.

For subtrees T of ${}^{<\kappa}\kappa$, the approximations[1] $\mathcal{G}_U(T, t)$ of the game $\mathcal{G}_\kappa(U, t)$ can be defined similarly.

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[1] In the sense of T. Hyttinen. Games and infinitary languages. *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes*, (64):1–32, 1987

Suppose U is a tree without κ-branches and $X \subseteq {}^{\kappa}\kappa$.

$$K_U(X) = \{x \in X : \mathbf{II} \text{ wins } \mathcal{V}_U(X,x)\};$$
$$S_U(X) = \{x \in X : \mathbf{I} \text{ wins } \mathcal{V}_U(X,x)\}.$$

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Let T be a subtree of ${}^{<\kappa}\kappa$.

$$K_U(T) = \{t \in T : \mathbf{II} \text{ wins } \mathcal{G}_U(T, t)\};$$
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Theorem (Sz.)

Let T be a subtree of ${}^{<\kappa}\kappa$ and let U be a tree without κ-branches.

1. $K_U([T]) \subseteq [K_U(T)]$.
2. $[T] - S_U([T]) \subseteq [T - S_U(T)]$.
3. *If κ has the tree property and T is a κ-tree, then*
$$K_U([T]) = [K_U(T)].$$

Thank you!