The tree property at $\aleph_{\omega+2}$

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The tree property

Recall that an uncountable regular cardinal κ has the tree property (TP(κ)) if every κ-tree has a cofinal branch.
In this talk we show the basic steps behind the proof of the following theorem:

Theorem (Friedman, Honzik, S. (2018))

(GCH) Suppose $0 \leq n < \omega$ is a natural number and there is κ which is $H(\lambda^{+n})$-hypermeasurable where λ is the least weakly compact above κ, then there is a forcing extension where the following hold:

1. $\kappa = \aleph_\omega$ *is strong limit and* $2^{\aleph_\omega} = \aleph_{\omega+n+2}$.
2. TP($\aleph_{\omega+2}$).

The continuum function at $\aleph_\omega$

Recall that if $\aleph_\omega$ is strong limit, then by a result by Shelah,

$$2^{\aleph_\omega} < \min(\aleph_{2^{\omega+}}, \aleph_{\omega_4}),$$

so we cannot aim for an arbitrary infinite gap.

We will mention at the end some open question, in particular whether we can extend our result to a countable gap with $\mathsf{TP}(\aleph_{\omega+2})$.

Forcing we will use

We use Mitchell forcing because it is more suitable to manipulate the continuum function.

- The product of the Mitchell and the Cohen forcing works nicely because Mitchell projects to the Cohen forcing (at relevant cardinals).
- The Mitchell forcing $\mathbb{M}(\kappa, \lambda)$ can be easily modify to force $2^\kappa > \kappa^{++}$ while forcing $\mathsf{TP}(\kappa^{++})$.

Mitchell forcing

Assume $\kappa < \lambda$ are infinite regular cardinals, with λ being inaccessible (weakly compact for us).

Definition

A condition in $\mathbb{M}(\kappa, \lambda)$ is a pair (p, q) such that p is a condition in $\mathrm{Add}(\kappa, \lambda)$ and q is a function with domain of size at most κ, $\mathrm{Dom}(q) \subseteq \lambda$, such that for all $\alpha \in \mathrm{Dom}(q)$, $q(\alpha)$ is an $\mathrm{Add}(\kappa, \alpha)$-name for a condition in $\mathrm{Add}(\kappa^+, 1)^{V^{\mathrm{Add}(\kappa,\alpha)}}$.

The ordering is defined on the next slide.

Mitchell forcing, the ordering

Assume $\kappa < \lambda$ are infinite regular cardinals, with λ being inaccessible. Then

Definition

A condition (p, q) is stronger than (p', q') if

(i) $p \leq p'$,

(ii) $\mathrm{dom}(q) \supseteq \mathrm{dom}(q')$ and for every $\beta \in \mathrm{dom}(q')$, $p \restriction \beta \Vdash_{\mathrm{Add}(\kappa,\beta)} q(\beta) \leq q'(\beta)$.

Mitchell forcing, basic properties

Assuming that $\kappa^{<\kappa} = \kappa$ and $\lambda > \kappa$ is an inaccessible cardinal, Mitchell forcing $\mathbb{M}(\kappa, \lambda)$ satisfies following:

- It is λ-Knaster and κ-closed.
- It collapses the cardinals in the open interval (κ^+, λ) to κ^+.
- It forces $2^\kappa = \lambda = \kappa^{++}$.

There is a projection from $\mathbb{M}(\kappa, \lambda)$ to $\mathrm{Add}(\kappa, \lambda)$.

The preservation of κ^+ is shown by the existence of a projection from $\mathrm{Add}(\kappa, \lambda) \times \mathbb{T}$ to $\mathbb{M}(\kappa, \lambda)$, where $\mathbb{T}$ is a κ^+-closed forcing (it has conditions of the form $(0, q)$ in $\mathbb{M}(\kappa, \lambda)$).[1]

The natural projection from $\mathbb{M}(\kappa, \lambda)$ to $\mathbb{M}(\kappa, \alpha)$ for $\kappa < \alpha < \lambda$, makes it possible to treat $\mathbb{M}(\kappa, \lambda)$ as an iteration, and write $\mathbb{M}(\kappa, \alpha) * \dot{\mathbb{R}}$.

[1] We call $\mathbb{T}$ the *term forcing*.

Branch lemmas

Let κ, λ be regular cardinals.

- (essentially Baumgartner) Assume that $\mathbb{P} \times \mathbb{P}$ is a κ-cc forcing notion. If T is a tree of height κ, then forcing with $\mathbb{P}$ does not add cofinal branches to T.
- (essentially Silver) Let $\kappa < \lambda$, with $2^\kappa \geq \lambda$. Assume that $\mathbb{P}$ is a κ^+-closed forcing notion. If T is a λ-tree, then forcing with $\mathbb{P}$ does not add cofinal branches to T.

The main strategy of the proof, with gap 3 (n=1)

- We prepare the universe V so that forcing $2^\kappa = \lambda^+$ with the Cohen forcing will preserve the measurability of κ (with some work, this is possible to do with the large-cardinal assumption that κ has a (κ, λ^+)-extender; supercompactness is not necessary[2]).
  - The preparation actually destroys the strong-limitness of λ. Thus λ is not weakly compact in the rest of the argument. This presents a technical obstacle which needs to be overcome.
- We use a variant of the Mitchell forcing $\mathbb{M} = \mathbb{M}(\kappa, \lambda, \lambda^+)$ to force 2^κ to be equal to λ^+, and simultaneously collapse cardinals in the interval (κ, λ).

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[2] See R. Honzik, Laver-like indestructibility for hypermeasurable cardinals, to appear in Arch. M. Log.

The main strategy of the proof, with gap 3

- In $V[\mathbb{M}]$, the tree property holds at κ^{++}, κ is still measurable, and we can define a Prikry forcing with collapses. Our final forcing is

$$\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathbb{Q},$$

  where $\mathbb{Q}$ is the Prikry forcing with collapses (defined with respect to some guiding generic).
- Now, the quotient analysis is much harder because of the Prikry forcing (with collapses).

Let give a brief review of the quotient analysis on the next slide.

The quotient analysis

- Let $k : V \to M$ be an elementary embedding with critical point λ. With the right setup we can write

$$k(\mathbb{M} * \mathbb{Q}) = (\mathbb{M} * \mathbb{Q}) * \mathbb{R},$$

  where $\mathbb{R}$ is the quotient forcing $k(\mathbb{M} * \mathbb{Q})/(\mathbb{M} * \mathbb{Q})$. In particular, if $G * x$ is $\mathbb{M} * \mathbb{Q}$-generic, then

$$\mathbb{R} = \{(p', q', r') \in k(\mathbb{M} * \mathbb{Q}) \,|\, (p', q', r') || k\text{''}(G * x)\}.$$

- We wish to show that $\mathbb{R}$ does not add branches to λ-trees over $M[G][x]$.

The quotient analysis

- Unlike the classical case (just with $\mathbb{M}$), it is not clear whether $\mathbb{R}$ regularly embeds into a product $P_1 \times P_2$, where $P_1 \times P_2$ is κ^+-cc and P_2 is κ^+-closed, which would make the argument simpler.
- Instead we will show directly that $\mathbb{R}$ does not add branches, which requires a careful analysis of when $(p, q, \dot{r})$ in $\mathbb{M} * \mathbb{Q}$ forces (p', q', r') in $k(\mathbb{M} * \mathbb{Q})$ into (or out of) $\dot{\mathbb{R}}$.

A rough outline of the argument is given on next slide.

- Suppose for contradiction T is a λ-tree in $V[\mathbb{M} * \mathbb{Q}]$ and $\mathbb{R}$ forces that $\dot{b}$ is a new branch in T. Let $\dot{T}$ be a $\mathbb{Q}$-name over $\mathbb{M}$ for T.
- We build a labelled tree $\mathcal{T}$ of height κ of conditions $a = (r, (p', q', r'))$ in $\mathbb{Q} * \mathbb{R}$ such that r decides how $\dot{T}$ looks locally and the whole a decide how $\dot{b}$ looks locally. In particular if $(r, (p', q', r'))$ decides a segment of $\dot{b}$ through $\dot{T}$, and for instance knows $y <_{\dot{T}} z$ are in $\dot{b}$, then already r knows $y <_{\dot{T}} z$.
- Since $\mathcal{T}$ has 2^κ cofinal branches, there are two branches v, w through $\mathcal{T}$ and respective conditions a_v and a_w which decide $\dot{b}|\delta$ the same way, say y (where δ is a level of $\dot{T}$ such that $\dot{b}|\delta$ is being decided by branches through $\mathcal{T}$).
- Continuing above these conditions, we get two more conditions which decide a restriction of $\dot{b}$ above y differently.

- By reflection (which is built into $\mathcal{T}$), such a difference is by necessity reflected down to some level $\delta' < \delta$ which contradicts the fact that a, b decide the restriction $b|\delta = y$ the same way.
- The construction of $\mathcal{T}$ and the subsequent arguments use crucially the fact that we work with a dense subforcing of $\mathbb{Q} * \mathbb{R}$ in which the conditions $(r, (p', q', r'))$ are such that $r \in \mathbb{Q}$ and $r' \in k(\mathbb{Q})$ have the same stem.
- With conditions from this dense subforcing, one can extend p' and q' more easily without running the risk of incompatibility with the stem of r (which would result in falling out of the quotient $\mathbb{R}$). Then we use the nice chain conditon of "p"-conditions and nice closure of the "q"-conditions (with respect to the term ordering) to build $\mathcal{T}$.

A variant of this argument is used to show any finite gap with $TP(\aleph_{\omega+2})$. In this variant, we essentially reduce the general case to the gap 3 case.

Open questions

Open questions:

1. Is it consistent to have an infinite gap with $\mathrm{TP}(\aleph_{\omega+2})$?
2. Can we in addition control other cardinal invariants besides $\mathfrak{c}(\aleph_\omega)$? For instance $\mathfrak{c}(\aleph_n)$ for $n < \omega$, or $\mathfrak{u}(\aleph_\omega)$?