Any $<\kappa$-closed forcing adding a dominating κ-real adds a κ-Cohen real.

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Main result

Theorem (K-Koelbing-Laguzzi-Wohofsky)

Any $<\kappa$-closed forcing adding a dominating κ-real adds a κ-Cohen real.

Some background

In the classical setting $\kappa = \omega$, **Cohen forcing** adds Cohen reals but no dominating reals, and **Laver forcing** adds dominating but no Cohen reals.

Definition

- $x \leq^* y$ (y dominates x) iff $\forall^\infty n \, (x(n) < y(n))$
- d is a dominating real over M if d dominates every real in M.

$\mathfrak{b}$ and $\mathrm{cov}(\mathcal{M})$

In the language of cardinal invariants:

- iterated Cohen forcing gives consistency of $\mathfrak{b} < \mathrm{cov}(\mathcal{M})$.
- iterated Laver forcing gives consistency of $\mathrm{cov}(\mathcal{M}) < \mathfrak{b}$.

Definition

- $\mathfrak{b}$ is the least size of an $F \subseteq \omega^\omega$ which **cannot** be dominated by a single $x \in \omega^\omega$.
- $\mathrm{cov}(\mathcal{M})$ is the least size of a family $\{X_\alpha \mid \alpha < \gamma\}$ such that $\bigcup_{\alpha<\gamma} X_\alpha = \omega^\omega$.

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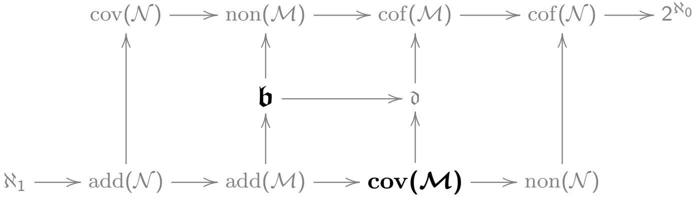
to increase	*use a forcing which*
$\mathfrak{b}$	adds **dominating** reals
$\mathrm{cov}(\mathcal{M})$	adds **Cohen** reals

Cichoń's diagram



Generalized Baire spaces

Generalizing the Cichoń diagram to the context of uncountable κ is one of the ongoing open projects in the study of generalized Baire spaces.

Dominating reals, Cohen reals, $\mathfrak{b}$ and $\text{cov}(\mathcal{M})$ all have straightforward generalizations.

Definition

- For $x, y \in \kappa^\kappa$, $x \leq^* y$ iff $\exists \alpha_0 < \kappa \; \forall \alpha > \alpha_0 \; (x(\alpha) < y(\alpha))$.
- $\mathcal{M}_\kappa$ is the ideal of κ-**meager** sets, i.e., κ-unions of nowhere dense sets.
- κ-**Cohen forcing** $\mathbb{C}_\kappa$ is the forcing with basic open conditions $\{[\sigma] \mid \sigma \in \kappa^{<\kappa}\}$ ordered by inclusion.
- $\mathfrak{b}_\kappa$ and $\text{cov}(\mathcal{M}_\kappa)$ as usual.

$\mathfrak{b}_\kappa$ and $\mathrm{cov}(\mathcal{M}_\kappa)$

κ-Cohen forcing does not add dominating κ-reals. Therefore $\mathrm{Con}(\mathfrak{b}_\kappa < \mathrm{cov}(\mathcal{M}_\kappa))$.

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Question

Is it consistent that $\mathrm{cov}(\mathcal{M}_\kappa) < \mathfrak{b}_\kappa$?

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Specifically, this would hold if we had a forcing adding dominating κ-reals but no Cohen κ-reals.

Rumour

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$\mathrm{Con}(\mathrm{cov}(\mathcal{M}_\kappa) < \mathfrak{b}_\kappa)$ was proved by Shelah et al., but with a different method: starting from a model of $\kappa^+ < \mathfrak{b}_\kappa = 2^\kappa$ and adding a witness for $\mathrm{cov}(\mathcal{M}_\kappa)$ of size κ^+.

Main result

Theorem (K-Koelbing-Laguzzi-Wohofsky)

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The most natural forcing to do this would be a generalization of **Laver forcing**.

First we prove a preliminary result: any suitable generalization of Laver forcing necessarily adds a κ-Cohen real. Then we use this to prove the main theorem.

Laver forcing

Definition

A **Laver tree** is a $T \subseteq \omega^{<\omega}$ such that for all $\sigma \in T$ extending stem(T), the set $\mathrm{succ}_T(\sigma)$ is infinite. $\mathbb{L}$ is the forcing consisting of Laver trees ordered by inclusion.

In the classical case, $\mathbb{L}$ adds a dominating real but satisfies the so-called "Laver property", which is preserved by iterations and implies that no Cohen reals are added. Hence, if $V \models$ CH then $V^{\mathbb{L}_{\omega_2}} \models \mathrm{cov}(\mathcal{M}) < \mathfrak{b}$.

Generalizing Laver forcing

How do you generalize Laver forcing to the κ^κ-setting?

Definition

A κ-**Laver tree** is a tree $T \subseteq \kappa^{<\kappa}$ which is

1. **limit-closed** (if $\{\sigma_\alpha : \alpha < \kappa\} \subseteq T$ is an increasing sequence of nodes, then $\bigcup_{\alpha<\kappa} \sigma_\alpha \in T$), and
2. for all $\sigma \in T$ extending stem(T), $|\mathbf{succ}_{\boldsymbol{T}}\boldsymbol{(\sigma)}| = \boldsymbol{\kappa}$.

Let $\mathbb{L}_\kappa$ denote the set of such trees ordered by inclusion.

Laver trees and Cohen reals

$\mathbb{L}_\kappa$ itself is a bit useless (e.g., we can show that it adds new subsets of ω), but one could consider other forcings whose conditions are Laver trees, e.g., by requiring that $\mathrm{succ}_T(\sigma)$ has additional properties (contains a club, is contained in a measure U on κ, etc.)

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$\mathbb{L}_\kappa$ itself is a bit useless (e.g., we can show that it adds new subsets of ω), but one could consider other forcings whose conditions are Laver trees, e.g., by requiring that $\mathrm{succ}_T(\sigma)$ has additional properties (contains a club, is contained in a measure U on κ, etc.)

However, **any** such partial order is going to add a κ-Cohen real.

Theorem (K-Koelbing-Laguzzi-Wohofsky)

Let $\mathbb{P} \subseteq \mathbb{L}_\kappa$ *be* **any** *partial order closed under the following condition:*

$$T \in \mathbb{P},\ \sigma \in T \quad \Rightarrow \quad T{\uparrow}\sigma := \{\tau \in T : \sigma \subseteq \tau \vee \tau \subseteq \sigma\} \in \mathbb{P}.$$

Then $\mathbb{P}$ *adds a* κ*-Cohen real.*

Supremum game

Definition

Let S be a stationary subset of $\mathrm{Cof}_\omega(\kappa) = \{\alpha > \kappa \ : \ \mathrm{cf}(\alpha) = \omega\}$. The **supremum game** $G^{sup}(S)$ is:

I	A_0		A_1		$\dots$	
II		β_0		β_1		$\dots$

- $A_n \subseteq \kappa$ with $|A_n| = \kappa$,
- $\beta_n \in A_n$.
- Player II wins iff $\sup\{\beta_n : n < \omega\} \in S$.

Supremum game

I	A_0		A_1		$\dots$	
II		β_0		β_1		$\dots$

Lemma

Let $S \subseteq \mathrm{Cof}_\omega(\kappa)$ be any stationary set. Then Player I *does* **not** *have a winning strategy in $G^{sup}(S)$.*

Supremum game

Proof.

Let σ be a strategy for Player I. Let $M \prec H_\theta$ be elementary for sufficiently large θ, such that $|M| < \kappa$, $\sigma \in M$ and $\delta := \sup(M \cap \kappa) \in S$.

We can do this because $\{\sup(M \cap \kappa) \; : \; M \prec H_\theta \wedge |M| < \kappa \wedge \sigma \in M\}$ contains a club.

Note that $\mathrm{cf}(\delta) = \omega$. Choose $\{\gamma_n : n < \omega\}$ cofinal in δ with $\gamma_n \in M$ for all n.

At each step n, inductively assume all A_k and β_k for $k < n$ are in M and let $A_n := \sigma(A_0, \beta_0, \ldots, \beta_{n-1})$. Since $\sigma \in M$, $A_n \in M$. Notice that H_θ satisfies the following statement:

$$\exists \beta > \gamma_n \, (\beta \in A_n).$$

By elementarity, it is also true in M, so let $\beta_n \in M$ be such.

This gives a sequence $\langle \beta_n : n < \omega \rangle$ with $\beta_n \in M \cap \kappa$ for all n, so $\sup_n \{\beta_n : n < \omega\} \leq \delta$.
On the other hand, since $\gamma_n \leq \delta_n$ for all n, we have
$\sup_n \{\beta_n : n < \omega\} \geq \sup_n \{\gamma_n : n < \omega\} = \delta$. So $\sup_n \{\beta_n : n < \omega\} = \delta \in S$, and so σ was not winning for Player I. □

Short Laver trees

Definition

A **short κ-Laver tree** is a tree $T \subseteq \kappa^{<\omega}$ such that

$$\forall \sigma \in T\ (|\mathrm{succ}_T(\sigma)| = \kappa).$$

Corollary

Let $S \subseteq \mathrm{Cof}_\omega(\kappa)$ *be stationary. If* T *is a short* κ*-Laver tree, then* $\exists x \in [T]$ *such that* $\sup\{x(n) : n < \omega\} \in S$.

Proof.

The tree T defines a strategy for Player I in the game $G^{sup}(S)$, which cannot be winning by the previous Lemma. □

Laver trees add Cohen reals

Let $S_0 \cup S_1$ be a stationary/co-stationary partition of $\mathrm{Cof}_\omega(\kappa)$. Define the mapping $\varphi : \kappa^\kappa \to 2^\kappa$ by

$$\varphi(x)(\alpha) := \begin{cases} 1 & \text{if } \sup\{x(\omega \cdot \alpha + n) : n < \omega\} \in S_0 \\ 0 & \text{if } \sup\{x(\omega \cdot \alpha + n) : n < \omega\} \notin S_1 \end{cases}$$

in other words:

> *"cut x up into κ-many ω-chunks, and map each chunk $x{\restriction}[\lambda, \lambda + \omega)$ to 1 or 0 depending on whether the supremum is in S_0 or S_1."*

Proof of Theorem

Theorem (K-Koelbing-Laguzzi-Wohofsky)

Let $\mathbb{P} \subseteq \mathbb{L}_\kappa$ be **any** *forcing closed under the following condition:*

$$T \in \mathbb{P},\ \sigma \in T \quad \Rightarrow \quad T{\uparrow}\sigma \in \mathbb{P}.$$

Then $\mathbb{P}$ adds a κ-Cohen real.

Proof of theorem.

Let $x_G \in \kappa^\kappa$ be the generic κ-real added by $\mathbb{P}$ (obtained by $\bigcup\{\text{stem}(T) \ : \ T \in G\}$). We show that $\varphi(x_G)$ is κ-Cohen.

Let $T \in \mathbb{P}$ be arbitrary and D dense in κ-Cohen. Let $\sigma := \text{stem}(T)$. Let $s := \varphi(\sigma)$ and let $t \in D$ extend s. By repeatedly applying the previous Corollary, we can find $\tau \in T$ such that $\sigma \subseteq \tau$ and $\varphi(\tau) = t$. By the closure assumption on $\mathbb{P}$, it follows that $T{\uparrow}\tau \in \mathbb{P}$ and clearly $T{\uparrow}\tau \Vdash \varphi(\dot{x}_G) \in [t] \in D$.

Thus $\varphi(x_G)$ is Cohen. □

A stronger result

We can actually improve the above result a bit.

Definition

A tree $T \subseteq \kappa^{<\kappa}$ is called a **pseudo-κ-Laver tree** if it is limit-closed and has the following property: $\forall \sigma \in T \; \exists \tau \in T$ s.t. $\sigma \subseteq \tau$ and $T \restriction [|\tau|, |\tau| + \omega)$ is a short κ-Laver tree. We use $\mathrm{p}\mathbb{L}_\kappa$ to denote the partial order of pseudo-κ-Laver trees ordered by inclusion.

A stronger result

Theorem (K-Koelbing-Laguzzi-Wohofsky)

Let $\mathbb{P} \subseteq \mathrm{p}\mathbb{L}_\kappa$ be **any** *forcing closed under the following condition:*

$$T \in \mathbb{P},\ \sigma \in T \quad \Rightarrow \quad T{\uparrow}\sigma \in \mathbb{P}.$$

Then $\mathbb{P}$ adds a κ-Cohen real.

Proof.

Similar as before, but instead we consider a stationary partition $\{S_t : t \in \kappa^{<\kappa}\}$ of $\mathrm{Cof}_\omega(\kappa)$. We can do this because $\kappa^{<\kappa} = \kappa$.

Use the mapping $\pi : \kappa^\kappa \to 2^\kappa$ given by $\pi(x) := t_0{}^\frown t_1{}^\frown t_2{}^\frown \dots$, where for all $\alpha < \kappa$, t_α is such that $\sup\{x(\alpha \cdot \omega + n) \ : \ n < \omega\} \in S_{t_\alpha}$.

Then, to force that $\pi(x_G)$ is Cohen, one only needs to extend the stem **once**, so that the supremum is contained in the appropriate S_t. □

Back to the main result

Definition

For $f : \kappa^{<\kappa} \to \kappa$ and $x \in \kappa^\kappa$, we say that ***x* strongly dominates *f*** if

$$\exists \alpha \, \forall \beta > \alpha \, (x(\beta) > f(x \restriction \beta)).$$

If M is a model of set theory, then x is **strongly dominating over *M*** if for all $f : \kappa^{<\kappa} \to \kappa$ with $f \in M$, x strongly dominates f.

If x is strongly dominating over M then x is dominating over M, but not vice versa. However:

Dominating vs. strongly dominating

Lemma

Let $M \models \kappa^{<\kappa} = \kappa$. If there is a dominating real over M, then there is a strongly dominating real over M.

Proof.

Let $\{\sigma_i : i < \kappa\}$ enumerate $\kappa^{<\kappa}$ in M, and we write $\ulcorner \sigma \urcorner = i$ iff $\sigma = \sigma_i$.

If d is dominating over M, inductively define $x(\alpha) := d(\ulcorner x{\restriction}\alpha \urcorner)$. We claim that x is strongly dominating over M.

Let $f : \kappa^{<\kappa} \to \kappa$ be in M, then z defined by $z(i) := f(\sigma_i)$ is also in M. Hence, for all but $<\kappa$-many i we have $z(i) < d(i)$. Hence, for all but $<\kappa$-many α we have $x(\alpha) = d(\ulcorner x{\restriction}\alpha \urcorner) > z(\ulcorner x{\restriction}\alpha \urcorner) = f(x{\restriction}\alpha)$. □

Interpretation structure

Let $\mathbb{P}$ be a $<\kappa$-closed forcing and $\dot{x}$ a $\mathbb{P}$-name for an element of κ^κ.

Definition

The **$\dot{x}$-decision structure** is the collection:

$$U_{\dot{x}} := \{(\sigma, q) \ : \ q \Vdash \sigma \subseteq \dot{x}\}$$

ordered by $(\sigma, q) \trianglelefteq (\tau, r)$ iff $\sigma \subseteq \tau$ and $r \leq q$.

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ordered by $(\sigma, q) \trianglelefteq (\tau, r)$ iff $\sigma \subseteq \tau$ and $r \leq q$.

Note: $U_{\dot{x}}$ is not a tree. However, we call $(\sigma, q) \in U_{\dot{x}}$ **κ-splitting** if there are $\{\xi_\alpha \ : \alpha < \kappa\}$ and corresponding $\{q_\alpha \ : \ \alpha < \kappa\}$ such that $(\sigma^\frown\langle\xi_\alpha\rangle, q_\alpha) \in U_{\dot{x}}$ and $(\sigma, q) \trianglelefteq (\sigma^\frown\langle\xi_\alpha\rangle, q_\alpha)$ for all α. Otherwise, we say that (σ, q) is **$<\kappa$-splitting**.

Strongly dominating reals

Lemma

Suppose $\dot{x}$ is a name such that $p_0 \Vdash_{\mathbb{P}} \dot{x}$ is a strongly dominating κ-real. Then for every $p \leq p_0$ there is $(\sigma, q) \in U_{\dot{x}}$ with $q \leq p$, such that all $(\tau, r) \in U_{\dot{x}}$ with $(\sigma, q) \trianglelefteq (\tau, r)$ are κ-splitting.

Proof.

Suppose not, and fix a counterexample $p \leq p_0$. Define $f : \kappa^{<\kappa} \to \kappa$ thus: for every $\tau \in \kappa^{<\kappa}$, **if there exists** $r \in \mathbb{P}$ such that $(\tau, r) \in U_{\dot{x}}$ and is $<\kappa$-splitting, then let $f(\tau) := \delta$ where δ is the an upper bound. For all other τ, let $f(\tau) := 0$.

Since $p \Vdash \dot{x}$ is strongly dominating, there exists $p' \leq p$, and an ordinal β_0, such that

$$p' \Vdash \forall \alpha > \beta_0 \, (\dot{x}(\alpha) > f(\dot{x} \restriction \alpha)) \qquad (*)$$

Let $q \leq p'$ decide $\dot{x} \restriction \beta_0$, which is possible because $\mathbb{P}$ is $<\kappa$-closed. In other words, there is σ with $|\sigma| \geq \beta_0$ such that $(q, \sigma) \in U_{\dot{x}}$. By assumption there exists $(\tau, r) \in U_{\dot{x}}$, such that $(\sigma, q) \trianglelefteq (\tau, r)$, and (τ, r) is $<\kappa$-splitting.

Strongly dominating reals

Lemma

Suppose $\dot{x}$ is a name such that $p_0 \Vdash_{\mathbb{P}} \dot{x}$ is a strongly dominating κ-real. Then for every $p \leq p_0$ there is $(\sigma, q) \in U_{\dot{x}}$ with $q \leq p$, such that all $(\tau, r) \in U_{\dot{x}}$ with $(\sigma, q) \trianglelefteq (\tau, r)$ are κ-splitting.

Proof.

So there exists δ which is larger than all $\xi < \kappa$ for which $\exists s \leq r\ (\tau^\frown\langle\xi\rangle, s) \in U_{\dot{x}}$, and $f(\tau) = \delta$. In other words: for all $s \leq r$, if $s \Vdash \tau^\frown\langle\xi\rangle \subseteq \dot{x}$ then $\xi < \delta$, so in fact $r \Vdash \dot{x}(\alpha) < \delta = f(\tau)$, where $\alpha = |\tau|$. Since also $r \Vdash \tau = \dot{x}\restriction\alpha$, we have $r \Vdash \dot{x}(\alpha) < f(\dot{x}\restriction\alpha)$. But since $r \leq q \leq p'$ and $\alpha > \beta_0$, this contradicts $(*)$. □

Proof of Main Theorem

Proof of Main Theorem

Let $\mathbb{P}$ be a $<\kappa$-closed forcing adding a dominating κ-real. Then it also adds a strongly dominating κ-real. Let $\dot{x}$ be a name such that $p_0 \Vdash_{\mathbb{P}} \dot{x}$ is strongly dominating. Consider the mapping $\varphi : \kappa^\kappa \to 2^\kappa$ from before (defined using $S_0 \cup S_1 = \mathrm{Cof}_\omega(\kappa)$). We claim that $p_0 \Vdash \varphi(\dot{x})$ is κ-Cohen.

Let D be Cohen-dense and $p \leq p_0$. Using the previous Lemma, find a $(\sigma, q) \in U_{\dot{x}}$ such that $q \leq p$ and all $(\tau, r) \in U_{\dot{x}}$ extending (σ, q) are κ-splitting.

Inductively, build a κ-Laver tree L, with $\mathrm{stem}(L) = \sigma$, and for every $u \in L$ pick a **unique** $r_u \leq q$ such that $(u, r_u) \in U_{\dot{x}}$.

Proof of Main Theorem

Construction of L:

- $\sigma \in L$, and $r_\sigma = q$.
- if $u \in L$ and r_u given, choose a unique $r_{u^\frown\langle\xi\rangle} \leq r_u$ for each $\xi < \kappa$ witnessing the κ-splitting of (u, r_u), and add $u^\frown\langle\xi\rangle$ to L.
- for every increasing sequence $\{u_\alpha : \alpha < \lambda\} \subseteq L$, by assumption there is a corresponding increasing sequence of $\mathbb{P}$-conditions $\{r_{u_\alpha} : \alpha < \lambda\}$. By $<\kappa$-closure, there exists an extending condition r_λ such that $r_\lambda \Vdash \bigcup_{\alpha<\lambda} \sigma_\alpha \subseteq \dot{x}$. So we can add $\bigcup_{\alpha<\lambda} \sigma_\alpha$ to L.

But then, using the supremum game and going along the Laver tree L, we can find a $(u, r) \in U_{\dot{x}}$ such that $(q, \sigma) \trianglelefteq (u, r)$ and $\varphi(u) \in D$. Since $r \Vdash u \subseteq \dot{x}$, also $r \Vdash \varphi(u) \subseteq \varphi(\dot{x})$, as we had to show. So indeed $p_0 \Vdash \varphi(\dot{x})$ is Cohen. □

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Remark: If we use the stationary partition $\{S_t : t \in \kappa^{<\kappa}\} \subseteq \mathrm{Cof}_\omega(\kappa)$ and the corresponding mapping π, we only need to assume that $\mathbb{P}$ is σ-closed and $<\kappa$-distributive (then it is enough to build a **short** κ-Laver tree at the relevant location).

Thank you!

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