Compactness principles and the ultrafilter number

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By *compactness principles* we mean the wide variety of principles which may consistently hold at a successor cardinal μ^+, but typically imply that μ^+ is a large cardinal in some inner model. Classical examples are:

- *tree property*,
- *stationary reflection*,
- *the failure of the approachability property*.

We are interested in studying the iteractions of these principles with other combinatorial properties, in particular with the continuum function $\mathfrak{c}$ and other cardinal invariants.

Let us briefly review the definitions of some compactness principles:

- A regular cardinal μ has the *tree property*, $TP(\mu)$, if every μ-tree has a cofinal branch.
- A cardinal μ^+ satisfies the *stationary reflection*, $SR(\mu^+)$, if every stationary subset of $\mu^+ \cap \mathrm{cof}(< \mu)$ reflects at a point of uncountable cofinality.
- Let $AP(\mu^+)$ mean "μ^+ satisfies the approachability property". We will not define it, just say that $\neg AP(\mu^+)$ is strictly stronger than $\neg\Box^*_\mu$, and in particular implies there are no special μ^+-Aronszajn trees.

Since $\mu^{<\mu} = \mu$ suffices to construct a special μ^+-Aronszajn tree, $\neg AP(\mu^+)$ and $TP(\mu^+)$ both imply[1]

$$\mu^{<\mu} > \mu.$$

This condition holds trivially if μ is singular, so let μ be a successor cardinal.[2] μ is then a successor of a cardinal which we denote κ (to be consistent with the notation later on).

$TP(\kappa^{++})$ and $\neg AP(\kappa^{++})$ imply

$$\mathfrak{c}(\kappa) \geq \kappa^{++}.$$

It follows that the structure of cardinal invariants at κ may be non-trivial.

[1] $SR(\mu^+)$ is consistent with $\mu^{<\mu} = \mu$.

[2] We omit the situation when μ is weakly inaccessible for simplicity.

The effect of $TP(\kappa^{++})$ and $\neg AP(\kappa^{++})$ on the continuum function $\mathfrak{c}$ is limited to $\mathfrak{c}(\kappa) \geq \kappa^{++}$.

It is possible to show an Easton-like theorem for the reflection properties: for instance, the failure of weak square at every $\aleph_n$, $1 < n < \omega$, is consistent with an arbitrary $\mathfrak{c}$ which satisfies $\mathfrak{c}(\aleph_n) > \aleph_{n+1}$ for $n < \omega$ (Stejskalova and myself). Recent work of Stejskalova shows that this is the case also for the tree property.

One can also show that it is consistent that $\aleph_\omega$ is strong limit, $TP(\aleph_{\omega+2})$ holds, and $\mathfrak{c}(\aleph_\omega)$ is any fixed cardinal in the interval $[\aleph_{\omega+2}, \aleph_{\omega+\omega})$ (Friedman, Stejskalova and myself).

Do other cardinal invariants at κ exhibit similar degree of freedom as $\mathfrak{c}(\kappa)$ with respect to compactness principles at κ^{++}?

This question has not been studied in detail. Some configurations of cardinal invariants are ensured using the standard methods which force compactness principles (Mitchell forcing, Sacks forcing, etc.). However, other configurations might be harder to obtain (and perhaps not possible at all).

We will introduce a general form of a forcing notion with several parameters in which the parameters can be chosen to obtain the desired configurations of some cardinal invariants, along with the compactness principles.

For concreteness, we will focus on the ultrafilter number $\mathfrak{u}(\kappa)$.[3]

Remark. Recall that the iteration of Sacks forcing at ω forces $TP(\aleph_2)$ with $\mathfrak{c}(\omega) = \aleph_2$, and if there were selective ultrafilters in V, then also $\mathfrak{u} = \aleph_1$ holds in the generic extension. It is known that selective ultrafilters on uncountable κ are normal measures. However, the argument with Sacks forcing does not immediately generalize because it is consistent that a single Sacks forcing at κ kills measurability of κ over "bad" models.

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[3] Supercompact cardinals will appear in our theorems: in some sense this is natural in the sense that ω is "absolutely" indestructible: supercompacts can be "Laver"-indestructible, recreating a part of the comfortable picture at ω. Weaker indestructibility notions are available for weaker large cardinals, and they can actually be sufficient (see [Hon] for strong cardinals, which may be enough).

Theorem (work in progress)

Suppose $\kappa < \lambda$ are supercompact cardinals. Then there is a generic extension in which κ remains supercompact, $\mathfrak{c}(\kappa)$ is arbitrarily large, TP(κ^{++}) holds, and $\mathfrak{u}(\kappa) = \kappa^+$.

Theorem (work in progress)

Suppose $\kappa < \lambda$ are supercompact cardinals. Then there is a generic extension in which κ is a strong limit singular cardinal with countable cofinality, $\mathfrak{c}(\kappa)$ is arbitrarily large, TP(κ^{++}) holds and $\mathfrak{u}(\kappa) = \kappa^+$.

Both theorems are proved using a forcing notion $\mathbb{P}^*(\mathbb{P}, \kappa, \lambda, \delta)$ which is built over an iteration $\mathbb{P}$ of length δ which ensures the small $\mathfrak{u}(\kappa)$ and was known before ($\mathbb{P}$ is treated as a parameter).

Forcing notion $\mathbb{P}^*$

The forcing notion $\mathbb{P}^* = \mathbb{P}^*(\mathbb{P}, \kappa, \lambda, \delta)$ has several parameters:

- $\omega \le \kappa < \lambda \le \delta$, κ is regular, λ is a large cardinal, δ is an ordinal of cofinality $> \kappa$.
- $\mathbb{P} = \mathbb{P}_\delta = \langle (\mathbb{P}_\alpha, \dot{Q}_\alpha) \,|\, \alpha < \delta \rangle$ is an iteration of length δ with each Q_α adding new subsets of κ and which has a strong form of κ^+-cc (at least to imply that $\mathbb{P} \times \mathbb{P}$ is κ^+-cc) and is typically κ-closed.

Forcing notion $\mathbb{P}^*$

Definition

Let $\mathbb{P}$ be as above, and let $A \subseteq \lambda$ be cofinal subset of λ which is sufficiently sparse.[a] Then $\mathbb{P}^*$ is a forcing notion with conditions (p, q) such that:

- $p \in \mathbb{P}$,
- q is function of size at most κ with its domain included in A such that for every $\alpha \in \mathrm{dom}(q)$, $q(\alpha)$ is a $\mathbb{P}_\alpha$-name for a condition in $\mathrm{Add}(\kappa^+, 1)^{V[\mathbb{P}_\alpha]}$.

The ordering is: $(p, q) \leq (p', q')$ iff $p \leq_{\mathbb{P}} p'$ and the domain of q extends the domain of q' and for all $\alpha \in \mathrm{dom}(q')$,

$$p|\alpha \Vdash_{\mathbb{P}_\alpha} q(\alpha) \leq q'(\alpha).$$

[a] See next slides.

With the right setup, $\mathbb{P}^*$ will force $TP(\lambda)$ and turn λ to κ^{++} (while collapsing cardinals in the interval (κ^+, λ)).

Let $k : V \to M$ be an elementary embedding with critical point λ. The key step in showing $TP(\lambda)$ is to argue that the quotient forcing

$$k(\mathbb{P}^*)/\mathbb{P}^*$$

does not add cofinal branches to λ-trees living in $M[\mathbb{P}^*]$. The complexity of the quotient analysis depends on the nature of $\mathbb{P}$, and also on the fact whether some extra forcings after $\mathbb{P}^*$ are involved (such as the Prikry forcing which may come after $\mathbb{P}^*$).

Examples

- The original Mitchell forcing is $\mathbb{P}^*$ for the following: κ regular, $\lambda > \kappa$ weakly compact, $\delta = \lambda$, and $\mathbb{P} = \mathsf{Add}(\kappa, \lambda)$. It forces the tree property at $\lambda = \kappa^{++}$ with $\mathfrak{c}(\kappa) = \lambda$.
- More generally, if $\delta \geq \lambda$ is a cardinal with cofinality $> \kappa$, and $\mathbb{P} = \mathsf{Add}(\kappa, \delta)$, the forcing $\mathbb{P}^*$ forces the same things as the Mitchell forcing but with $\mathfrak{c}(\kappa) = \delta$.

More generalizations

- In Abraham's paper [Abr83], which obtains simultaneously $TP(\aleph_2)$ and $TP(\aleph_3)$, the relevant $\mathbb{P}^*$ is obtained by modifying *the second coordinate* "q" in the Mitchell forcing: instead of using $\mathrm{Add}(\kappa^+, 1)$ for collapsing, Abraham uses a Laver function to guess a κ^+-directed closed forcing which is used later on. The associated $\mathbb{P}^*$ forces $TP(\aleph_2)$ and $\mathfrak{c}(\aleph_0) = \aleph_2$.

More generalizations

- In a paper by Cummings and Foreman [CF98], in which they obtain $TP(\kappa^{++})$ at a singular strong limit cardinal κ with countable cofinality, $\mathbb{P}$ is a more general forcing of length λ (which will become κ^{++}): it is built up of a system of forcing notions $\text{Add}(\kappa, \alpha) * \dot{\mathbb{Q}}_{U_\alpha}$ which commute under the regular embedding relation, and in which $\dot{\mathbb{Q}}_{U_\alpha}$ is the Prikry forcing with respect to some U_α obtained by reflecting from a measure U in $V[\text{Add}(\kappa, \lambda) * \dot{\mathbb{Q}}_U]$.

  The associated $\mathbb{P}^*$ forces $TP(\kappa^{++})$ with $\mathfrak{c}(\kappa) = \kappa^{++}$.[4]

[4] The analysis is more involved in this case because the quotient $\text{Add}(\kappa, \lambda) * \dot{Q}_U / \text{Add}(\kappa, \alpha) * \dot{Q}_{U_\alpha}$ does not immediately possess a strong κ^+-cc condition (unlike the usual iterations $\mathbb{P}$ whose initial segments force the tail to have the same properties).

Regarding the requirement on the sparsity of A:

- If $\delta = \lambda$, A can usually be whole of λ. This is true for instance for the original Mitchell forcing.
- In Abraham's paper on $TP(\aleph_2) + TP(\aleph_3)$, [Abr83], A is composed of successor cardinals below λ.
- For the purposes of this paper, consider the following A: let $\ell : \lambda \to \lambda$ be a Laver function and let A be the set of all *singular closure points* of ℓ.

  Then for any $\delta > \lambda$, one can choose a supercompact embedding $k : V \to M$ with critical point λ such that $k(\ell)(\lambda) = \delta$ and consequently $k(A)$ avoids the interval $[\lambda, \delta)$. This enables us to analyse the quotient forcing $k(\mathbb{P}^*)/\mathbb{P}^*$ more easily.

Properties of $\mathbb{P}^*$

Let $\mathbb{T}$ be the forcing notion composed of all pairs $(1_{\mathbb{P}}, q) \in \mathbb{P}^*$, where $1_{\mathbb{P}}$ is the weakest condition in $\mathbb{P}$, with the ordering induced from $\mathbb{P}^*$. Then the following hold:

- By our assumptions $\mathbb{P}$ is κ^+-cc, and is κ-closed, and so does not collapse cardinals.
- $\mathbb{T}$ is κ^+-closed.
- $\mathbb{P}$ is regularly embedded into $\mathbb{P}^*$.
- There is projection from $\mathbb{P} \times \mathbb{T}$ to $\mathbb{P}^*$; it follows

$$\mathbb{P}^* \equiv \mathbb{P}^* * \mathbb{P}^*/\mathbb{P},$$

  where the quotient forcing $\mathbb{P}^*/\mathbb{P}$ is forced to be κ^+-distributive.

With λ being inaccessible, $\mathbb{P}^*$ is typically λ-cc, and since each $\dot{Q}_\alpha$ adds new subsets of κ, $\mathbb{P}^*$ collapses exactly the cardinals in the interval (κ^+, λ), and forces $\mathfrak{c}(\kappa) = |\delta|^V$.

First theorem

Recall the first theorem:

Theorem (work in progress)

Suppose $\kappa < \lambda$ are supercompact cardinals. Then there is a generic extension in which κ remains supercompact, $\mathfrak{c}(\kappa)$ is arbitrarily large, TP(κ^{++}) holds, and $\mathfrak{u}(\kappa) = \kappa^{+}$.

First theorem

For the first theorem, $\mathbb{P}$ can be the forcing from either of the two following papers:

- *Universal graphs at the successor of a singular cardinal* [DS03] (JSL, 68, 2003), in which Džamonja and Shelah developed a method for obtaining a model where κ is inaccessible (in fact supercompact) and $\kappa^+ \ll \mathfrak{c}(\kappa)$ and $\mathfrak{u}(\kappa) = \kappa^+$.[5]
- The method of construction was later simplified by Brooke-Taylor, Fischer, Friedman and Montoya in *Cardinal characteristics at κ in a small $\mathfrak{u}(\kappa)$ model* (APAL, 168, 2017, [BTFFM17]), and the value of $\mathfrak{u}(\kappa)$ was shown to consistently have any reasonable value below $\mathfrak{c}(\kappa)$.

[5] It was not the main purpose of the paper so the result is not stated explicitly in the paper.

We will not give a detailed definition of $\mathbb{P}$, but state the important properties:

- $\mathbb{P}$ is is an iteration of length μ^+, $\lambda \leq \mu$, in which $\dot{Q}_\alpha$, for $\alpha < \mu^+$, is a lottery among all Mathias forcings at κ defined with respect to normal measures $\dot{U}_\alpha$ in $V[\mathbb{P}_\alpha]$. It follows that below a condition $p \in \mathbb{P}$ which chooses the measures on a proper initial segment δ of μ^+, the forcing $\mathbb{P}_\delta \downarrow p$ is κ^+-Knaster (with the usual cardinal-arithmetic assumptions).
- By a Löwenheim-Skolem argument, a suitable $\delta \in (\mu, \mu^+)$ of cofinality κ^+ is found so that a carefully chosen $\dot{U}$ in $V[\mathbb{P}]$ reflects at a cofinal set I in δ of order type κ^+, and below a certain condition p, $\mathbb{P}_\delta \downarrow p$ forces that $\dot{U}$ is generated in $V[\mathbb{P}_\delta \downarrow p]$ by the Mathias κ-reals on I.

First theorem

In order to prove our theorem, we need to check two things:

1. We know that $\mathfrak{u}(\kappa) = \kappa^+$ in $V[\mathbb{P}_\delta \downarrow p]$ for a suitable δ of cofinality κ^+ and $p \in \mathbb{P}$ (note that $\mathbb{P}_\delta \downarrow p$ is κ^+-Knaster). Does this remain to be true in $V[\mathbb{P}^*_\delta \downarrow p]$?
2. We need to show that $\mathbb{P}^*_\delta \downarrow p$ forces $TP(\lambda)$.

Using the fact that the quotient forcing $\mathbb{P}^*_\delta \downarrow p / \mathbb{P}_\delta \downarrow p$ is forced to be κ^+-distributive, the same measure which is a witness for $\mathfrak{u}(\kappa) = \kappa^+$ in $V[\mathbb{P}^*_\delta \downarrow p]$ is a witness for $\mathfrak{u}(\kappa) = \kappa^+$ in $V[\mathbb{P}_\delta \downarrow p]$.

First theorem, (2)

Let $\mathbb{P} = \mathbb{P}_\delta \downarrow p$ and $\mathbb{P}^* = \mathbb{P}^*_\delta \downarrow p$ for simplicity of notation. Let $k : V \to M$ be a supercompact embedding[6] with critical point λ and $k(\ell)(\lambda) = \delta$. Since the domain of the q-coordinates in $\mathbb{P}^*$ is sufficiently sparse, we obtain

$$k(\mathbb{P}^*) \equiv \mathbb{P}^*_\lambda * k(\mathbb{P})_{[\lambda,\delta)} * \mathsf{Tail} \equiv \mathbb{P}^* * \mathsf{Tail}$$

and

$$k|\mathbb{P}^* : \mathbb{P}^* \to k(\mathbb{P}^*) \text{ is a regular embedding,}$$

as the forcing $\mathbb{P}^*$ is λ-cc, and hence the k-image of a maximal antichain is the pointwise image.

[6] A weakly compact embedding suffices if $\delta < \lambda^+$. For larger δ, a stronger embedding is required.

First theorem, (2)

Since k is not the identity on $\mathbb{P}^*$ – because of $\delta > \lambda$, a bit more involved analysis is required to finish the argument in the standard fashion: Suppose G' is $k(\mathbb{P}^*)$-generic over V, and let $G' = G * H$ be $\mathbb{P}^* *$ Tail-generic, where $G = k^{-1}{}''\, G'$. It follows we can lift

$$k : V[G] \to M[G'].$$

Using the closure properties of M, $k^{-1}{}''\, G' = G$ is an element of $M[G']$, and so $M[G'] = M[G][H]$, and we can write

$$k : V[G] \to M[G][H].$$

First theorem, (2)

Any λ-Aronszajn tree T in $M[G]$[7] has a cofinal branch in $M[G][H]$, but this is shown to be impossible since using a product analysis, the Tail forcing is regularly embedded into a product of a κ^+-Knaster and a κ^+-closed forcing – and such forcings do not add cofinal branches to λ-trees if we have $2^\kappa \geq \lambda$ in $M[G]$.

[7] Importantly, $\mathbb{P}$ is a non-homogeneous iteration of length δ, so it is not clear whether an initial segment of $\mathbb{P}^*$ below λ^+ adds a λ-Aronszajn tree if the whole iteration does: so we need to deal with the whole iteration of length δ which in general requires λ to be larger than a weakly compact cardinal.

Second theorem

Let us recall the second theorem:

Theorem (work in progress)

Suppose $\kappa < \lambda$ are supercompact cardinals. Then there is a generic extension in which κ is a strong limit singular cardinal with countable cofinality, $\mathfrak{c}(\kappa)$ is arbitrarily large, TP(κ^{++}) holds and $\mathfrak{u}(\kappa) = \kappa^{+}$.

Second theorem

We will leave the details to another talk,[8] here are the bare essentials:

- The $\mathbb{P}$ comes from a paper by Garti and Shelah [GS12]. It is an iteration which makes sure that certain products have true cofinality κ^+.
- The final forcing is

$$\mathbb{P}^*(\mathbb{P}, \kappa, \lambda, \delta) * \mathbb{Q}_U,$$

  where $\mathbb{Q}_U$ is the Prikry forcing with respect to some normal measure U on κ.

The addition of the Prikry forcing makes the quotient analysis for the tree property more complicated, but a method used in a paper by Friedman, Stejskalova and myself [FHS18] seems suitable.

[8] Probably in Vienna in September 10-14, 2018.

Open questions and further work

- Can the second theorem be modified to have other cofinalities besides ω with small $\mathfrak{u}(\kappa)$? In the paper [GS12], Garti and Shelah claim this is possible using the Magidor (Radin) forcing. It is probable this can be combined with $TP(\kappa^{++})$, but it is open.
- Can the second theorem be modified to include collapses in $\mathbb{Q}_U$ to have $\kappa = \aleph_\omega$ (or $\aleph_{\omega^2}$) in the final model? This is probably open even for the $\mathfrak{u}(\kappa) = \kappa^+$, and the more so with the reflection.
- Is it possible to have small $\mathfrak{u}(\kappa)$ at a successor κ?

- The forcing $\mathbb{P}$ used in the first theorem actually controls many other cardinal invariants. It seems probable that $\mathbb{P}^*$ may achieve the same things. In general, can every known cardinal invariant pattern be recreated with the reflection properties? On a single cardinal, but also on a segment of cardinals?
- Still more generally, is there an Easton-like theorem which controls the cardinal invariants in the presence of compactness principles?

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