Classical Theory of Cardinal Characteristics

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22 August, 2018

Outline

- Combinatorics
- Topology and Measure
- Well-Behaved Sets
- Models of Set Theory

Baire Space

Baire space is the set ${}^\omega\omega$ of infinite sequences of natural numbers.

Topologize it as the product of copies of the discrete space ω.

It carries the product measure induced by the measure on ω that gives each $\{n\}$ measure $1/2^{n+1}$.

It maps continuously into the power set $\mathcal{P}(\omega)$ or equivalently into ${}^\omega 2$ by

$$(a_0, a_1, \dots) \mapsto \underbrace{1 \dots 1}_{a_0} 0 \underbrace{1 \dots 1}_{a_1} 0 \dots .$$

It maps continuously onto the real interval $[0, 1)$.

The preceding maps are fairly well-behaved.

(It maps homeomorphically onto $\mathbb{R} - \mathbb{Q}$ by continued fractions.)

A Few Combinatorial Cardinals

For $f, g \in {}^\omega\omega$, define $f \leq^* g$ to mean that $f(n) \leq g(n)$ for all but finitely many $n \in \omega$.

The *dominating number* $\mathfrak{d}$ is the smallest cardinality of a *dominating* family $\mathcal{D} \subseteq {}^\omega\omega$, i.e.,

$$(\forall f \in {}^\omega\omega)(\exists g \in \mathcal{D})\, f \leq^* g.$$

The *bounding number* $\mathfrak{b}$ is the smallest cardinality of an *unbounded* family $\mathcal{B} \subseteq {}^\omega\omega$, i.e.,

$$(\forall f \in {}^\omega\omega)(\exists g \in \mathcal{D})\, g \not\leq^* f.$$

Then

$$\aleph_1 \leq \mathrm{cf}(\mathfrak{b}) = \mathfrak{b} \leq \mathrm{cf}(\mathfrak{d}) \leq \mathfrak{d} \leq \mathfrak{c}.$$

Any cardinals satisfying these constraints can consistently be $\mathfrak{b}$ and $\mathfrak{d}$.

Splitting and Unsplitting

The *splitting number* $\mathfrak{s}$ is the smallest cardinality of a *splitting family* $\mathcal{S} \subseteq \mathcal{P}(\omega)$, i.e.

$$(\forall X \in [\omega]^\omega)(\exists S \in \mathcal{S})\,\text{Both } X \cap S \text{ and } X - S \text{ are infinite.}$$

Terminology: S *splits* X.

The *unsplitting number* $\mathfrak{r}$ is the smallest cardinality of a family $\mathcal{R} \subseteq [\omega]^\omega$ such that no single subset of ω splits all members of $\mathcal{R}$.

Then $\mathfrak{s} \leq \mathfrak{d}$ and $\mathfrak{r} \geq \mathfrak{b}$.

These two inequalities have the same proof.

A Format for Cardinals

Many cardinal characteristics are naturally defined in the form "The smallest cardinality of a family $\mathcal{X}$ such that, for every y, there is an $x \in \mathcal{X}$ suitably related to y."

Usually, the relevant x's and y's are reals and "suitably related" has a "nice" definition.

$$\|(A_-, A_+, A)\| = \min\{|\mathcal{X}| : \mathcal{X} \subseteq A_+ \text{ and } (\forall y \in A_-)(\exists x \in \mathcal{X})\,(y, x) \in A\}.$$

(When the sets $A_\pm$ and the relation A are definable in second-order arithmetic, such characteristics are a special case of Zapletal's *tame* characteristics. He showed (assuming large cardinals) that many characteristics $\mathfrak{y}$ have a "canonical" forcing to increase them, such that the only tame characteristics increased by that forcing are the ones provably $\geq \mathfrak{y}$.)

A Format for Proofs

A *morphism* from (A_-, A_+, A) to (B_-, B_+, B) is a pair of functions $\tau_- : B_- \to A_-$ and $\tau_+ : A_+ \to B_+$ such that, for all $b \in B_-$ and $a \in A_+$,

$$(\tau_-(b), a) \in A \implies (b, \tau_+(a)) \in B.$$

The existence of such a morphism implies

$$\|(A_-, A_+, A)\| \geq \|(B_-, B_+, B)\|.$$

The inequalities $\mathfrak{s} \leq \mathfrak{d}$ and $\mathfrak{r} \geq \mathfrak{b}$ (and also the trivial $\mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$) arise from morphisms.

Duality

The *dual* of (A_-, A_+, A) is $(A_+, A_-, \neg\breve{A})$, where $\breve{A}$ means the converse of A.

$\mathfrak{b}$ is the dual of $\mathfrak{d}$.
$\mathfrak{r}$ is the dual of $\mathfrak{s}$.

A morphism from (A_-, A_+, A) to (B_-, B_+, B) is essentially the same as a morphism between the duals in the other direction, from $(B_+, B_-, \neg\breve{B})$ to $(A_+, A_-, \neg\breve{A})$.

This is the sense in which $\mathfrak{s} \leq \mathfrak{d}$ and $\mathfrak{r} \geq \mathfrak{b}$ have the same proof.

Baire Space, Cantor Space, and the Reals

To express a given topological space X as a union of compact subsets, how many compact subsets do we need?

For the Cantor Space, 1 (since the space is itself compact).

Same for [0,1].

For $\mathbb{R}$, $\aleph_0$, and likewise for $[0, 1)$.

For Baire space ${}^{\omega}\omega$, $\mathfrak{d}$.

Nevertheless, for many purposes, these spaces are interchangeable.

Baire Category and Lebesgue Measure

Two notions of “small” sets and “well-behaved” sets.

Baire Category: A set is *meager* if it can be covered by countably many closed sets whose interiors are empty.

A set has the *Baire property* if it differs from an open set by a meager set.

Lebesgue Measure: A set *has measure zero* if it can be covered by a set of intervals of arbitrarily small total length.

A set is *Lebesgue measurable* if it differs from a G_δ set by a set of measure zero.

Similarity Between Category and Measure

All Borel sets have the Baire property and are Lebesgue measurable.

The same is true of analytic sets (continuous images of Borel sets) and their complements.

But ZFC does not prove the same for continuous images of complements of analytic sets.

ZFC plus large cardinals proves the same for the whole projective hierarchy (and more).

Many more category-measure similarities. "Measure and Category: A Survey of the Analogies between Topological and Measure Spaces" by John Oxtoby.

Dissimilarity Between Category and Measure

ZF + DC + "All sets of reals are Lebesgue measurable" is consistent relative to ZFC + "There is an inaccessible cardinal".
ZF + DC + "All sets of reals have the Baire property" is consistent relative to just ZFC.

For any cardinal κ, if every union of κ sets of Lebesgue measure zero has Lebesgue measure zero, then every union of κ meager sets is meager.

The converse is not provable in ZFC.

"Set Theory: On the Structure of the Real Line" by Tomek Bartoszyński and Haim Judah.

Cichoń's Diagram

$$\begin{array}{ccccccc}
\mathbf{cov}(\mathcal{L}) & \longleftarrow & \mathbf{non}(\mathcal{B}) & \longleftarrow & \mathbf{cof}(\mathcal{B}) & \longleftarrow & \mathbf{cof}(\mathcal{L}) \\
\Big| & & \downarrow & & \downarrow & & \Big| \\
\Big| & & \mathfrak{b} & \longleftarrow & \mathfrak{d} & & \Big| \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbf{add}(\mathcal{L}) & \longleftarrow & \mathbf{add}(\mathcal{B}) & \longleftarrow & \mathbf{cov}(\mathcal{B}) & \longleftarrow & \mathbf{non}(\mathcal{L})
\end{array}$$

A Game

Given a set $X \subseteq {}^{\omega}\omega$, define a game as follows:
Players I and II alternately choose finite, nonempty sequences of natural numbers.
Player I wins a play if the concatenation of all the chosen sequences (in order) is in X; otherwise II wins.

Player I has a winning strategy iff X is comeager in some interval.
Player II has a winning strategy iff X is meager.

Existence of enough strategies (determinacy of enough games) implies the Baire property.

Another Game

This time, let $X \subseteq {}^{\omega}2$. The players alternate moves. Player I always plays finite sequences of 0's and 1's; Player II plays, at each move, a single 0 or 1. Again, Player I wins if the concatenation of all the moves is in X and Player II wins otherwise.

Player I has a wining strategy iff X has a perfect subset.
Player II has a winning strategy iff X is countable.

Determinacy

ZFC proves that games with Borel payoff sets are determined. (Martin)

This gives alternative proofs that, for example, all Borel sets have the Baire property and, if uncountable, have perfect subsets (and therefore have cardinality $\mathfrak{c}$).

Analytic games need not be determined, but they are if a measurable cardinal exists. (Martin)

Even larger cardinals imply determinacy for projective games and beyond.

Determinacy for *all* subsets of ${}^{\omega}\omega$ contradicts the axiom of choice but is consistent with ZF+DC (if large cardinals are consistent). It implies regularity properties for all sets of reals, and a rich theory of cardinal arithmetic.

The Constructible Universe

Gödel's constructible universe L, originally introduced to prove the relative consistency of AC and GCH, has important additional properties relevant to Baire space.

Most importantly, it has a Δ^1_2 good well-ordering of the reals.

As a result, constructions that are usually done using the axiom of choice can be done *definably* in L.

For example, L satisifes "There exists a Δ^1_2 non-principal ultrafilter on ω," "There exists a Δ^1_2 Hamel basis for the reals over the rationals," "There exists a Δ^1_2 maximal almost disjoint family of subsets of ω," and many similar statements.

(Sometimes, with extra work, one can improve Δ^1_2 to Π^1_1 here.)

The Lévy-Solovay Model

If κ is an inaccessible cardinal, *Lévy forcing* below κ collapses all cardinals $< \kappa$ to ω, so that κ is the $\aleph_1$ of the forcing extension.

Solovay showed that, in this model, every set of reals that is definable using ordinals and reals (or even countable sequences of ordinals) as parameters is well-behaved:

It has the Baire property.

It is Lebesgue measurable.

It either has a perfect subset or is countable.

So in the submodel HOD($\mathbb{R}$) of this forcing extension, all sets of reals are well-behaved in these senses. AC fails but DC holds.

Lebesgue Density Theorem

Let λ denote Lebesgue measure, and say that "almost all" reals have some property if the set of reals lacking the property has measure zero.

If $X \subseteq \mathbb{R}$ is Lebesgue measurable, then for almost all $x \in X$,

$$\frac{\lambda(X \cap (x-\varepsilon, x+\varepsilon))}{2\varepsilon} \to 1 \quad \text{as } \varepsilon \to 0.$$