Silver dichotomy for countable cofinalities

Vincenzo Dimonte

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Joint work with Xianghui Shi

Previously...

When λ is a strong limit cardinal of cofinality ω, descriptive set theory can be done in ${}^{\lambda}2$, or equivalently in ${}^{\omega}\lambda$, $\Pi_{n\in\omega}\lambda_n$ or $V_{\lambda+1}$.

Many results in classical descriptive set theory hold also in this setting.

In general, the results that are dependent to some tree-structure generalize very well.

$\mathrm{I0}(\lambda)$ has an influence on this setting in the same way that AD has an influence on classical descriptive set theory.

Theorem (Silver, 1993)

Let X be a Polish space and $E \subseteq X^2$ be a coanalytic equivalence relation on X. Then exactly one of the following holds:

- E has at most countably many classes;
- there is a continuous injection $\varphi : {}^{\omega}2 \to X$ such that for distinct $x, y \in {}^{\omega}2$ $\neg\varphi(x)E\varphi(y)$.

Is this true also for the generalized Baire space?

Theorem (Friedman, Kulikov 2014)

Suppose $V = L$ and κ inaccessible. Then the order $\langle \mathcal{P}(\kappa), \subset \rangle$ can be embedded into the set of Borel equivalence relations on 2^κ strictly below the identity, ordered with Borel reducibility.

Theorem (Silver, 1993)

Let E be a coanalytic equivalence relation on ${}^{\omega}2$. Then exactly one of the following holds:

- E has at most countably many classes;
- there is a continuous injection $\varphi : 2^{\omega} \to {}^{\omega}2$ such that for distinct $x, y \in 2^{\omega}$ $\neg\varphi(x)E\varphi(y)$.

Theorem?

Let E be a coanalytic equivalence relation on $\prod_{n\in\omega}\lambda_n$. Then exactly one of the following holds:

- E has at most countably many classes;
- there is a continuous injection $\varphi : \prod_{n\in\omega}\lambda_n \to \prod_{n\in\omega}\lambda_n$ such that for distinct $x, y \in \prod_{n\in\omega}\lambda_n$ $\neg\varphi(x)E\varphi(y)$.

Theorem?

Let E be a coanalytic equivalence relation on $\prod_{n\in\omega}\lambda_n$. Then exactly one of the following holds:

- E has at most λ many classes;
- there is a continuous injection $\varphi : \prod_{n\in\omega}\lambda_n \to \prod_{n\in\omega}\lambda_n$ such that for distinct $x, y \in \prod_{n\in\omega}\lambda_n$ $\neg\varphi(x)E\varphi(y)$.

Theorem! (D.-Shi)

Let λ_n be measurable cardinals. Let E be a coanalytic equivalence relation on $\Pi_{n\in\omega}\lambda_n$. Then exactly one of the following holds:

- E has at most λ many classes;
- there is a continuous injection $\varphi : \Pi_{n\in\omega}\lambda_n \to \Pi_{n\in\omega}\lambda_n$ such that for distinct $x, y \in \Pi_{n\in\omega}\lambda_n$ $\neg\varphi(x)E\varphi(y)$.

"Definition"

Let E be an equivalence relation on some product space. We say that E has the "singleton property" if for all x, y, if they differ *only* in one coordinate, then $\neg xEy$.

Theorem (Shelah 1988)

If E is a co-analytic equivalence relation on ${}^{\omega}2$ with the singleton property, then there is a continuous injection $\varphi : {}^{\omega}2 \to {}^{\omega}2$ such that for distinct $x, y \in {}^{\omega}2$ $\neg\varphi(x)E\varphi(y)$.

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Let E be an equivalence relation on some product space. We say that E has the "singleton property" if for all x, y, if they differ *only* in one coordinate, then $\neg xEy$.

Theorem (Shelah 2003)

Let λ_n be measurable cardinals.If E is a co-analytic equivalence relation on $\prod_{n\in\omega}\lambda_n$ with the singleton property, then there is a continuous injection $\varphi : \prod_{n\in\omega}\lambda_n \to \prod_{n\in\omega}\lambda_n$ such that for distinct $x, y \in \prod_{n\in\omega}\lambda_n$ $\neg\varphi(x)E\varphi(y)$.

Fix a dense subset S of ${}^{<\omega}2$ that intersects every level in exactly one element. Let G_0 be the directed graph that couples two elements if they start with an element of S and differ only in the next coordinate.

Theorem (G_0-dichotomy)

Let G be an analytic directed graph on ${}^{\omega}2$. Then exactly one of the following holds:

- there is a (Borel) $\aleph_0$-colouring of G;
- there is a continuous function from ${}^{\omega}2$ to itself that is a homomorphism from G_0 to G.

This actually generalizes nicely, with almost the same proof.

Fix a dense subset S of $\prod_{n\in\omega}\lambda_n$ that intersects every level in exactly one element. Let G_0 be the directed graph that couples two elements if they start with an element of S and differ only in the next coordinate.

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Fix a dense subset S of $\Pi_{n\in\omega}\lambda_n$ that intersects every level n in exactly κ_{n-1} element. Let G_0 be the directed graph that couples two elements if they start with an element of S and differ only in the next coordinate.

Theorem?

Let G be an analytic directed graph on $\Pi_{n\in\omega}\lambda_n$. Then exactly one of the following holds:

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Theorem?

Let G be an analytic directed graph on $\Pi_{n\in\omega}\lambda_n$. Then exactly one of the following holds:

- there is a λ-colouring of G;
- there is a continuous function from $\Pi_{n\in\omega}\lambda_n$ to itself that is a homomorphism from G_0 to G.

This actually generalizes nicely, with almost the same proof.

Fix a dense subset S of $\prod_{n\in\omega}\lambda_n$ that intersects every level n in exactly κ_{n-1} element. Let G_0 be the directed graph that couples two elements if they start with an element of S and differ only in the next coordinate.

Theorem! (D.-Shi)

Let G be an analytic directed graph on $\prod_{n\in\omega}\lambda_n$. Then exactly one of the following holds:

- ~~there is a λ-colouring of G~~ (actually, something more complicated, but equivalent for graphs that are the complement of an equivalence relation);
- there is a continuous function from $\prod_{n\in\omega}\lambda_n$ to itself that is a homomorphism from G_0 to G.

This actually generalizes nicely, with almost the same proof.

Now, let E be a co-analytic equivalence relation on $\Pi_{n\in\omega}\lambda_n$. Then its complement G is an analytic directed graph, therefore either E has $\leq \lambda$ equivalence classes, or there is a continuous function $\varphi : \Pi_{n\in\omega}\lambda_n \to \Pi_{n\in\omega}\lambda_n$ such that $x, y \in G_0$ iff $\neg\varphi(x)E\varphi(y)$. The problem is now that φ is possibly not injective.

Classically, from the G_0-dichotomy to Silver Dichotomy we use the meagre-comeagre structure of ${}^\omega 2$. This creates many problems in $\Pi_{n\in\omega}\lambda_n$, but Shelah's theorem can save us: the complement of G_0 has the singleton property, and we can use a similar argument to finally prove the Silver Dichotomy.

Can we get rid of the measurable cardinals?

Are measurable cardinals the key to understand the Baire structure of ${}^{\lambda}2$?

One of the main points of the Axiom of Determinacy is that it generalizes regularity properties for all subsets of reals. This is true also for Silver Dichotomy:

Theorem (AD)

Let E be an equivalence relation on ${}^{\omega}2$. Then exactly one of the following holds:

- the classes of E are well-ordered;
- there is a continuous injection $\varphi : {}^{\omega}2 \to {}^{\omega}2$ such that for distinct $x, y \in {}^{\omega}2$ $\neg\varphi(x)E\varphi(y)$.

One of the main points of I0 is that it generalizes AD-like results to higher cardinal. Does it work also in this case?

Open problem I0(λ)

Let E be an equivalence relation on ${}^{\lambda}2$. Is it true that exactly one of the following holds?

- the classes of E are well-ordered;
- there is a continuous injection $\varphi : {}^{\lambda}2 \to {}^{\lambda}2$ such that for distinct $x, y \in {}^{\lambda}2$ $\neg\varphi(x)E\varphi(y)$.

Forbidden slide 1 (not enough time)

Brief summary of proof of Shelah's result.

Consider the double diagonal Prikry forcing $\mathbb{P}$ that adds *two* Prikry sequences in λ. This forcing has two important characteristics:

- if M is a model of cardinality λ, then there is a M-generic set for $\mathbb{P}$ in V;
- only the tails of the generic are meaningful, so changing just one coordinate maintain the genericity.

Forbidden slide 2 (not enough time)

The fact that E is co-analytic is also important: this means that the formula that defines E is absolute between models that contain V_λ.

So the proof goes like this: pick M small model that contains everything. If there is a condition of $\mathbb{P}$ that forces that the two elements of the generic are E-related, then also those in V are E-related. Switching one coordinate we do the same, but this contradicts the singleton property or the fact that E is an equivalence relation.

Using generic absoluteness, we have a partial result:

Theorem

Suppose $I0(\lambda)$, as witness by j, and let $(\lambda_n)_{n\in\omega}$ be the critical sequence of j. Suppose that all subsets of $V_{\lambda+1}$ are $U(j)$-representable. Then if $E \in L(V_{\lambda+1})$ is an equivalence relation with the singleton property, there is a continuous injection $\Pi_{n\in\omega}\lambda_n \to \Pi_{n\in\omega}\lambda_n$ such that for distinct $x, y \in \Pi_{n\in\omega}\lambda_n$ $\neg\varphi(x)E\varphi(y)$.

Thanks for watching.