Completeness and universality for analytic equivalence relation

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Generalized Baire Spaces
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Notation

Generalized Descriptive Set Theory is the study of definable subsets of the Generalized Baire Spaces ${}^{\kappa}\kappa$, and of all theirs isomorphic spaces.

Basic definitions

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  - A function $f\colon X \to Y$ is κ^+-**Borel measurable** if the preimage of every open $U \subseteq Y$ is κ^+-Borel.
  - $f\colon X \to Y$ is a κ^+-**Borel isomorphism** if f^{-1} exists and is κ^+-Borel.
- A κ-space is **standard Borel** if it is κ^+-Borel isomorphic to a κ^+-Borel subset of ${}^\kappa\kappa$.

Analytic sets

Definition
A set $A \subseteq X$ is κ^+-**analytic** (or $\mathbf{\Sigma}^1_1$) if it is the continuous image of of a closed subset of ${}^{\kappa}\kappa$.

Generalized Borel reducibility

Definition

Let X and Y be standard Borel κ-space, and P, R be binary relations over X and Y, respectively. We say that P **Borel reduces** to R (or $\boldsymbol{P \leq_B R}$) if and only if there is a κ^+-Borel $f \colon X \to Y$ such that

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- The notion $\leq_B$ has been used successfully to analyze the complexity of $\boldsymbol{\Sigma}^1_1$ quasi-orders and equivalence relations.

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Definition

An equivalence relation E on a standard κ^+-Borel space X is a **complete analytic equivalence relation (CAER)** if

1. $E \subseteq X^2$ is $\mathbf{\Sigma}^1_1$;
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- The classification problem associated to a complete $\mathbf{\Sigma}^1_1$ equivalence relation is as complicated as it could be.
- While many results in GDST are independent from the model of set theory, a lot of results of completeness are derived from ZFC.

Some examples

Theorem (Mildenberger-Motto Ros)
The bi-embeddability relation $\equiv^{\kappa}_{\mathsf{GRAPHS}}$ *is a CAER.*

Theorem (C. 2018)
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- The second theorem was derived before establishing the completeness for $\equiv_{\mathsf{TFA}} = \equiv^{\omega}_{\mathsf{TFA}}$ in the classical framework.
- Now we know that $\equiv_{\mathsf{TFA}}$ is a CAER (C.-Thomas), but no explicit reduction from $\equiv_{\mathsf{GRAPHS}}$ to $\equiv_{\mathsf{TFA}}$ is known.

Combing through the literature...

Proposition

The bi-embeddability relation of κ-sized structure is a CAER in the following cases.

- Unital rings (ess. Fried, and Sichler 1973);
- Fields (ess. Fried, and Kollár 1982);
- Quandles and others (Brooke-Taylor, and S. Miller);
- ...

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invariantly universal if for every $\mathbf{\Sigma}^1_1$ equivalence relation E there is an $\mathcal{L}_{\kappa^+\kappa}$-sentence ψ such that $X_\psi \subseteq X_\varphi$ and $E \sim_B \equiv_\psi$.

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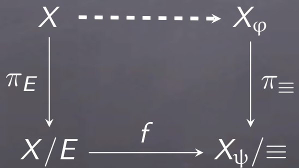
- I.e., there is a bijection between the quotient spaces $f\colon X/E \to X_\psi/\equiv_\varphi$ such that both f and f^{-1} admit Borel lifting.



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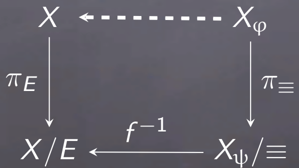

Strong universality

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- The methods generalizes for fields, quandles and other structures...

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such that for all unnested atomic K-formulæ $\phi(x_0, \ldots, x_n)$ and all $\bar{b} = (b_0, \ldots, b_n) \in \partial_\Gamma(B)$, we have

$$A \models \phi[f_\Gamma(b_0), \ldots, f_\Gamma(b_n)] \iff B \models \phi_\Gamma[b_0, \ldots, b_n].$$

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Lemma (C.-Motto Ros)

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Corollary

For every $\mathcal{K}$-formula $\phi(\bar{x})$ there is a formula $\phi_\Gamma(\bar{x})$ in the language of groups such that

$$G \models \phi[f_G(\bar{a})] \iff H(G) \models \phi_\Gamma[\bar{a}].$$

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✓ ...the inverse map has Borel lifting too. □