Special $\aleph_2$-Aronszajn trees and GCH

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The problem of building models of consequences, at the level of $H(\omega_2)$, of classical forcing axioms together with CH has a long history, starting with Jensen's landmark result that Suslin's Hypothesis is compatible with GCH.

Most of the work in the area done so far proceeds by showing that some suitable countable support iteration whose iterands are proper forcing notions not adding new reals fails to add new reals at limit stages.

There are (nontrivial) limitations to what can be achieved in this area. In fact, there cannot be any 'master' iteration lemma:

A.–Larson–Moore: Modulo a mild large cardinal assumption, there are two Π_2 statements over $H(\omega_2)$, each of which can be forced, using proper forcing, to hold together with CH, and whose conjunction implies $2^{\aleph_0} = 2^{\aleph_1}$.

Above result closely tied to the following concrete well–known obstacle to not adding reals: Given a ladder system $\vec{C} = (C_\delta \,:\, \delta \in Lim(\omega_1))$, let $\mathrm{Unif}(\vec{C})$ denote the statement that for every colouring $F : \mathrm{Lim}(\omega_1) \longrightarrow \{0, 1\}$ there is $G : \omega_1 \longrightarrow \{0, 1\}$ such that that for every $\delta \in \mathrm{Lim}(\omega_1)$ there is some $\alpha < \delta$ such that $G(\xi) = F(\delta)$ for all $\xi \in C_\delta \setminus \alpha$. We say that G uniformizes F on $\vec{C}$.

Given $\vec{C}$ and F as above there is a natural forcing notion, $\mathcal{Q}_{\vec{C},F}$, for adding a uniformizing function for F on $\vec{C}$ by initial segments. Easy to see that $\mathcal{Q}_{\vec{C},F}$ is proper, adds the intended uniformizing function, and does not add reals. However, any long enough iteration of forcings of the form $\mathcal{Q}_{\vec{C},F}$, even with a fixed $\vec{C}$, will necessarily add new reals. In fact, the existence of a ladder system $\vec{C}$ for which $\mathrm{Unif}(\vec{C})$ holds cannot be forced together with CH in any way whatsoever, as this statement actually implies $2^{\aleph_0} = 2^{\aleph_1}$ (Devlin–Shelah).

Proof: Fix a bijection $h : \omega \longrightarrow \omega \times \omega$ such that $i \leq n$ if $h(n+1) = (i, j)$. For each $g : \omega_1 \longrightarrow 2$ construct $f_n : \omega_1 \longrightarrow 2$ $(n < \omega)$ such that

$$f_0 = g$$

and

$$f_{n+1} \upharpoonright C_\delta =_{\text{fin}} f_i(\delta + j)$$

for every limit $\delta \neq 0$, where $h(n+1) = (i, j)$.
Given f_k $(k \leq n)$, f_{n+1} exists by applying $\text{Unif}(\vec{C})$ to the colouring

$$\delta \longrightarrow f_i(\delta + j)$$

But now, for each limit $\delta \neq 0$, $(f_n \upharpoonright \delta \ : \ n < \omega)$ determines $(f_n \upharpoonright \delta + \omega \ : \ n < \omega)$. Hence,

$$(f_n \upharpoonright \omega \ : \ n < \omega)$$

determines

$$(f_n \ : \ n < \omega),$$

and in particular $f_0 = g$. Hence $2^{\aleph_0} = 2^{\aleph_1}$. $\square$

Definition

Measuring holds if and only if for every sequence $\vec{C} = (C_\delta : \delta \in \omega_1)$, if each C_δ is a closed subset of δ in the order topology, then there is a club $C \subseteq \omega_1$ such that for every $\delta \in C$ there is some $\alpha < \delta$ such that either

- $(C \cap \delta) \setminus \alpha \subseteq C_\delta$, or
- $(C \setminus \alpha) \cap C_\delta = \emptyset$.

We say that C *measures* $\vec{C}$.

Measuring implies $\neg$WCG: Suppose $\vec{C} = (C_\delta \,:\, \delta \in \mathrm{Lim}(\omega_1))$ ladder system and $C \subseteq \omega_1$ is a club measuring $\vec{C}$. Then, for every $\delta \in C$, if δ is a limit point of limit points of C, then a tail of $C \cap \delta$ is disjoint from C_δ since $\mathrm{ot}(C_\delta) = \omega$.

Natural forcing for adding a club measuring a given $\vec{C}$ by initial segments is proper and adds no new reals. On the other hand it is not known if these forcings can (consistently) be iterated without adding new reals. Strongest failures of Club–Guessing known to be within reach of current techniques for iterating proper forcing without adding reals are in the region of $\neg$WCG (Shelah, NNR revisited).

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(Moore) Is Measuring consistent with CH?

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In joint work with Mota, we addressed Moore's question. In order to do so we distanced ourselves from the tradition of preserving CH by not adding reals; we aimed at building interesting models of CH by a cardinal–preserving forcing which actually adds reals (but only $\aleph_1$–many of them).

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Forcing with symmetric systems of models as side conditions

Finite–support forcing iterations involving symmetric systems of models as side conditions are useful in situations in which, for example, we want to force

- consequences of classical forcing axioms at the level of $H(\omega_2)$, together with
- $2^{\aleph_0}$ large.

Given a cardinal κ and $T \subseteq H(\kappa)$, a finite $\mathcal{N} \subseteq [H(\kappa)]^{\aleph_0}$ is a *T–symmetric system* if

(1) for every $N \in \mathcal{N}$,

$$(N, \in, T) \preccurlyeq (H(\kappa), \in, T),$$

(2) given N_0, $N_1 \in \mathcal{N}$, if $N_0 \cap \omega_1 = N_1 \cap \omega_1$, then there is a unique isomorphism

$$\Psi_{N_0,N_1} : (N_0, \in, T) \longrightarrow (N_1, \in, T)$$

and Ψ_{N_0,N_1} is the identity on $N_0 \cap N_1$.

(3) Given N_0, $N_1 \in \mathcal{N}$ such that $N_0 \cap \omega_1 = N_1 \cap \omega_1$ and $M \in N_0 \cap \mathcal{N}$, $\Psi_{N_0,N_1}(M) \in \mathcal{N}$.

(4) Given M, $N_0 \in \mathcal{N}$ such that $M \cap \omega_1 < N_0 \cap \omega_1$, there is some $N_1 \in \mathcal{N}$ such that $N_1 \cap \omega_1 = N_0 \cap \omega_1$ and $M \in N_1$.

The pure side condition forcing

$$\mathcal{P}_0 = (\{\mathcal{N} : \mathcal{N} \text{ a } T\text{–symmetric system}\}, \supseteq)$$

(for any fixed $T \subseteq H(\kappa)$) preserves CH:

This exploits the fact that given N, $N' \in \mathcal{N}$, $\mathcal{N}$ a symmetric system, if $N \cap \omega_1 = N' \cap \omega_1$, then $\Psi_{N,N'}$ is an isomorphism

$$\Psi_{N,N'} : (N; \in, \mathcal{N} \cap N) \longrightarrow (N'; \in, \mathcal{N} \cap N')$$

Proof: Suppose $(\dot{r}_\xi)_{\xi<\omega_2}$ are names for subsets of ω and $\mathcal{N} \Vdash_{\mathcal{P}_0} \dot{r}_\xi \neq \dot{r}_{\xi'}$ for all $\xi \neq \xi'$. For each ξ, let N_ξ be a sufficiently correct model such that $\mathcal{N}$, $\dot{r}_\xi \in N_\xi$.

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By CH we may find $\xi \neq \xi'$ such that there is an isomorphism

$$\Psi : (N_\xi; \in, T^*, \mathcal{N}, \dot{r}_\xi) \longrightarrow (N_{\xi'}; \in, T^*, \mathcal{N}, \dot{r}_{\xi'})$$

(where T^* is the satisfaction predicate for $(H(\kappa); \in, T)$). Then $\mathcal{N}^* = \mathcal{N} \cup \{N_\xi, N_{\xi'}\} \in \mathcal{P}_0$. But $\mathcal{N}^*$ is $(N_\xi, \mathcal{P}_0)$–generic and $(N_{\xi'}, \mathcal{P}_0)$–generic.

Now, let $n < \omega$ and let $\mathcal{N}'$ be an extension of $\mathcal{N}^*$. Suppose $\mathcal{N}' \Vdash_{\mathcal{P}_0} n \in \dot{r}_\xi$. Then there is $\mathcal{N}'' \in \mathcal{P}_0$ extending both $\mathcal{N}'$ and some $\mathcal{M} \in N_\xi \cap \mathcal{P}_0$ such that $\mathcal{M} \Vdash_{\mathcal{P}_0} n \in \dot{r}_\xi$. **By symmetry,** $\mathcal{N}''$ extends also $\Psi(\mathcal{M})$. But $\Psi(\mathcal{M}) \Vdash_{\mathcal{P}_0} n \in \Psi(\dot{r}_\xi) = \dot{r}_{\xi'}$.

We have shown $\mathcal{N}^* \Vdash_{\mathcal{P}_0} \dot{r}_\xi \subseteq \dot{r}_{\xi'}$, and similarly we can show $\mathcal{N}^* \Vdash_{\mathcal{P}_0} \dot{r}_{\xi'} \subseteq \dot{r}_\xi$. Contradiction since $\mathcal{N}^*$ extends $\mathcal{N}$ and $\xi \neq \xi'$.

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In typical forcing iterations with symmetric systems as side conditions, $2^{\aleph_0}$ is large in the final extension. Even if $\mathcal{P}_0$ can be seen as the first stage of these iterations, the forcing is in fact designed to add reals at (all) subsequent successor stages.

Something one may want to try at this point: Extend the symmetry requirements **also** to the working parts in such a way that the above CH–preservation argument goes trough. Hope to be able to force something interesting this way.

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Theorem

(A.–Mota) (CH) Let $\kappa > \omega_2$ be a regular cardinal such that $2^{<\kappa} = \kappa$. There is then a partial order $\mathcal{P}$ with the following properties.

(1) *$\mathcal{P}$ is proper and $\aleph_2$–Knaster.*

(2) *$\mathcal{P}$ forces the following statements.*

   (a) *Measuring*
   (b) *CH*
   (c) *$2^{\mu} = \kappa$ for every uncountable cardinal $\mu < \kappa$.*

The following question addresses whether or not adding reals is a necessary feature of forcing Measuring.

Question

(Moore) Does Measuring imply that there are non-constructible reals?

This is joint work with Mohammad Golshani.

Let κ be a regular uncountable cardinal.

- A κ–tree is a tree T of height κ all of whose levels are smaller than κ. A κ*–Aronszajn tree* is a κ–tree which has no κ–branches.

- A κ*–Souslin tree* is a κ–tree which has no κ–branches and no antichains of size κ.

- If $\kappa = \lambda^+$, a κ–Aronszajn tree T is said to be *special* if there exists a function $f : T \to \lambda$ such that $f(x) \neq f(y)$ whenever $x, y \in T$ are such that $x <_T y$. We say that f *specializes* T.

- The special Aronszajn tree property at $\kappa = \lambda^+$, $\mathrm{SATP}(\kappa)$, is the statement "there exist κ–Aronszajn trees and all such trees are special".

Aronszajn trees were introduced by Kurepa, and Aronszajn (1934) proved the existence, in ZFC, of a special $\aleph_1$–Aronszajn tree. Later, Specker (1949) showed that $2^{<\lambda} = \lambda$ implies the existence of special λ^+–Aronszajn trees for λ regular, and Jensen (1972) produced special λ^+–Aronszajn trees for singular λ in L.

Baumgartner, Malitz and Reinhardt (1970) showed that Martin's Axiom + $2^{\aleph_0} > \aleph_1$ implies SATP($\aleph_1$), and hence Souslin's Hypothesis at $\aleph_1$ as well. Later, and as already mentioned, Jensen (1974) produced a model of GCH in which SATP($\aleph_1$) holds.

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The situation at $\aleph_2$ turned out to be more complicated. Jensen (1972) proved that the existence of an $\aleph_2$–Souslin follows from each of the hypotheses $\mathsf{CH} + \diamondsuit(\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega_1\})$ and $\square_{\omega_1} + \diamondsuit(\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega\})$. The second result was improved by Gregory (1976); he proved that GCH together the existence of a non–reflecting stationary subset of $\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega\}$ yields the existence of an $\aleph_2$–Souslin tree.

Laver and Shelah (1981) produced, relative to the existence of a weakly compact cardinal, a model of $\mathsf{ZFC} + \mathsf{CH}$ in which $\mathsf{SATP}(\aleph_2)$ holds. But in their model $2^{\aleph_1} > \aleph_2$, and the following remained a major open problem (s. e.g. Kanamori–Magidor 1977):

Question

Is ZFC+GCH consistent with the non–existence of $\aleph_2$–Souslin trees?

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In December 2017, while visiting Golshani in Tehran, we started thinking about combining the ideas from Measuring + CH with the Laver–Shelah construction for SATP($\aleph_2$). We eventually succeeded:

Theorem (A.–Golshani) Suppose κ is a weakly compact cardinal. Then there exists a set–generic extension of the universe in which

(1) GCH holds,

(2) $\kappa = \aleph_2$, and

(3) SATP($\aleph_2$) holds (and hence there are no $\aleph_2$–Souslin trees).

(1) Our argument can be easily extended to the successor of any regular cardinal. Also, the method can be applied to the construction of a model of SATP($\aleph_1$) + GCH (no need of weakly compact cardinal for this). This consistency result is originally due to Shelah.

(2) Our large cardinal assumption is optimal:

- ⋆ Rinot (2017) proved that GCH + Souslin's Hypothesis at $\aleph_2$ implies $\neg\square(\omega_2)$; on the other hand, Todorčević (1987) proved that $\neg\square(\omega_2)$ implies that ω_2 is weakly compact in L.

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Definition of forcing

Let κ be weakly compact. W.l.o.g. we may assume $2^\mu = \mu^+$ for all $\mu \geq \kappa$.
We define by induction on β, a sequence $\langle \mathbb{Q}_\beta \mid \beta \leq \kappa^+ \rangle$ of forcing notions.

Fix $\beta \leq \kappa^+$ and suppose $\mathbb{Q}_\alpha$ defined for all $\alpha < \beta$.

The new idea: Revisionism (copying information from the future into the past).

Let us call

$$\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$$

an *edge below* β if

(1) for all $i \in \{0, 1\}$, γ_i is an ordinal in the closure of $N_i \cap \mathrm{Ord}$ such that $\gamma_i \leq \beta$ and $(N_i, \in, \Phi_\alpha) \preccurlyeq (H(\kappa^+), \in, \Phi_\alpha)$ for all $\alpha \in N_i \cap \gamma_i$ (for suitable sequence of increasingly expressible $\Phi_\alpha \subseteq H(\kappa^+)$), and

(2) $N_0 \cong N_1$ via an isomorphism $\Psi_{N_0,N_1} : N_0 \longrightarrow N_1$ such that
   - (i) $(N_0, \in, \Phi_\alpha) \cong (N_1, \in, \Phi_{\Psi_{N_0,N_1}(\alpha)})$ for all $\alpha < \gamma_0$ such that $\Psi_{N_0,N_1}(\alpha) < \gamma_1$,
   - (ii) Ψ_{N_0,N_1} is the identity on $N_0 \cap N_1$, and
   - (iii) $\Psi_{N_0,N_1}(\xi) \leq \xi$ for every ordinal $\xi \in N_0$.

We build $\mathbb{Q}_\beta$ as a forcing with side conditions consisting of sets of edges. Given an edge $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$ in the side condition, we copy information in N_0 attached to $\alpha < \gamma_0$ via Ψ_{N_0,N_1} into N_1 if $\Psi_{N_0,N_1}(\alpha) < \gamma_1$. **We do not require that information in N_1 attached to $\Psi_{N_0,N_1}(\alpha)$ be copied into N_0.**

A condition in $\mathbb{Q}_\beta$ is an ordered pair of the form $q = (f_q, \tau_q)$ with the following properties.

(1) f_q is a countable function such that $\mathrm{dom}(f_q) \subseteq \kappa^+ \cap \beta$ and such that the following holds for every $\alpha \in \mathrm{dom}(f_q)$.

  (a) If $\alpha = 0$, then $f_q(\alpha) \in \mathrm{Col}(\omega_1, {<}\kappa)$.
  (b) If $\alpha > 0$, then

$$f_q(\alpha) : \kappa \times \omega_1 \to \omega_1$$

  is a countable function.

(2) τ_q is a countable set of edges below β.

(3) The following holds for every edge $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_q$.

  (a) If $\langle (N_0', \gamma_0'), (N_1', \gamma_1') \rangle \in N_0 \cap \tau_q$, then

$$\langle (\Psi_{N_0,N_1}(N_0'), \gamma_0^*), (\Psi_{N_0,N_1}(N_1'), \gamma_1^*) \rangle \in \tau_q$$

  for 'sufficiently high' γ_0^* and γ_1^*.
  (b) The following holds for each nonzero ordinal $\alpha \in \mathrm{dom}(f_q) \cap N_0 \cap \gamma_0$ such that $\Psi_{N_0,N_1}(\alpha) < \gamma_1$.
    (i) $\Psi_{N_0,N_1}(\alpha) \in \mathrm{dom}(f_q)$
    (ii) $f_q(\alpha) \restriction \delta_{N_0} \times \omega_1 \subseteq f_q(\Psi_{N_0,N_1}(\alpha))$

(4) For all $\alpha < \beta$, $q \restriction \alpha \in \mathbb{Q}_\alpha$, where

$$q \restriction \alpha = (f_q \restriction \alpha, \tau_q \restriction \alpha)$$

(5) For every nonzero $\alpha \in \mathrm{dom}(f_q)$, if $x \neq y$ are both in $\mathrm{dom}(f_q(\alpha))$, and

$$(f_q(\alpha))(x) = (f_q(\alpha))(y),$$

then $q \restriction \alpha$ forces, in $\mathbb{Q}_\alpha$, that x and y are incomparable in $\underset{\sim}{T}_\alpha$.

The extension relation:
Given $q_1, q_0 \in \mathbb{Q}_\beta$, $q_1 \leq_\beta q_0$ (q_1 is an extension of q_0) if and only if the following holds.

(A) $\mathrm{dom}(f_{q_0}) \subseteq \mathrm{dom}(f_{q_1})$

(B) for every $\alpha \in \mathrm{dom}(f_{q_0})$, $f_{q_0}(\alpha) \subseteq f_{q_1}(\alpha)$.

(C) $\tau_{q_0} \subseteq \tau_{q_1}$

Main facts

(0) For every $\beta \leq \kappa^+$,

   (i) $\mathbb{Q}_\alpha \subseteq \mathbb{Q}_\beta$ for all $\alpha < \beta$, and
   (ii) if $\mathrm{cf}(\beta) \geq \kappa$, then $\mathbb{Q}_\beta = \bigcup_{\alpha<\beta} \mathbb{Q}_\alpha$.

(1) Thanks to the fact that we are only copying information 'from the future into the past', $(\mathbb{Q}_\beta)_{\beta \leq \kappa^+}$ is a forcing iteration (i.e., $\mathbb{Q}_\alpha \lessdot \mathbb{Q}_\beta$ for all $\alpha < \beta$): Given $q \in \mathbb{Q}_\beta$ and $r \in \mathbb{Q}_\alpha$, if $r \leq_\alpha q \restriction \alpha$, then

$$(f_r \cup f_q \restriction [\alpha, \beta), \tau_q \cup \tau_r)$$

is a common extension of q and r in $\mathbb{Q}_\beta$.

(2) $\mathbb{Q}_1$ forces $\kappa = \aleph_2$.

(3) $\mathbb{Q}_\beta$ is σ-closed for every $\beta \leq \kappa^+$. In fact, every decreasing ω-sequence of $\mathbb{Q}_\beta$-conditions has a greatest lower bound in $\mathbb{Q}_\beta$. In particular, forcing with $\mathbb{Q}_\beta$ does not add new ω-sequences of ordinals, and therefore it preserves both ω_1 and CH.

(4) $\mathbb{Q}_{\kappa^+}$ adds κ–many new subsets of ω_1, but not more than that; in particular, $\mathbb{Q}_{\kappa^+}$ preserves $2^{\aleph_1} = \aleph_2$ [essentially the same argument we saw a few slides back].

(5) If $\mathbb{Q}_{\kappa^+}$ is κ-c.c., then it forces SATP($\aleph_2$).

The κ–chain condition

Lemma

For each $\beta \leq \kappa^+$, $\mathbb{Q}_\beta$ has the κ-chain condition.

This is the only place where we use the weak compactness of κ.

Proof sketch: Let us fix a sequence $(q_\lambda \mid \lambda < \kappa)$ of $\mathbb{Q}_\beta$-conditions. Want $\lambda \neq \lambda'$ such that q_λ and $q_{\lambda'}$ are compatible. The proof is by induction on β.

The case $\beta = 0$ is trivial and the case $\beta = 1$ follows from inaccessibility of κ ($\mathbb{Q}_1$ is essentially the Lévy collapse turning κ into ω_2). The case $\beta = \kappa^+$ follows from Fact (0) together with the induction hypothesis.

It remains to consider the case $1 < \beta < \kappa^+$. For this case we use an (almost literal) adaptation of the following key separation argument from Laver–Shelah.

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It remains to consider the case $1 < \beta < \kappa^+$. For this case we use an (almost literal) adaptation of the following key separation argument from Laver–Shelah.

Lemma

(Laver–Shelah) Suppose κ *is weakly compact and* $(\mathcal{Q}_\beta)_{\beta \leq \eta}$ *is a countable support iteration such that* $\mathcal{Q}_1 = \mathrm{Col}(\omega_1, {<}\kappa)$ *and for all* $1 \leq \beta < \eta$, $\mathcal{Q}_{\beta+1} = \mathcal{Q}_\beta * \dot{\mathcal{R}}_\beta$, *where* $\dot{\mathcal{R}}_\beta$ *is the natural forcing for specializing*
some given κ*–Aronszajn tree* $\dot{T}_\beta$*. Then* $\mathcal{Q}_\beta$ *is* κ*–c.c. for all* $\beta \leq \eta$*.*

Proof sketch: Let $M \preccurlyeq H(\theta)$ containing everything relevant of size κ and such that ${}^{<\kappa}M \subseteq M$ and let $(M_\lambda)_{\lambda<\kappa}$ be a continuous filtration of M. Let $\mathbb{Q}^*_\alpha = \mathbb{Q}_\alpha \cap M$ for all α. By κ–c.c. of $\mathbb{Q}_\alpha$ for alll $\alpha < \eta$ (by induction hypothesis), $\mathbb{Q}^*_\alpha \lessdot \mathbb{Q}_\alpha$ for all $\alpha < \eta$.

Given conditions q^L, q^R, $\alpha \in \mathrm{dom}(f_{q^L}) \cap \mathrm{dom}(f_{q^R})$, $x \in \mathrm{dom}(f_{q^L}(\alpha))$ and $y \in \mathrm{dom}(f_{q^R}(\alpha))$ (x and y may or may not be equal), we say that

- *x and y are separated by $q^L \upharpoonright \alpha$ and $q^R \upharpoonright \alpha$ at stage α below λ by means of $\bar{x}$, $\bar{y}$*

if there is $\bar{\rho} < \lambda$, together with $\zeta \neq \zeta'$ in ω_1, such that letting $\bar{x} = (\bar{\rho}, \zeta)$ and $\bar{y} = (\bar{\rho}, \zeta')$,

$$q^L \upharpoonright \alpha \Vdash_\alpha \bar{x} <_{\dot{T}_\alpha} x$$

and

$$q^R \upharpoonright \alpha \Vdash_\alpha \bar{y} <_{\dot{T}_\alpha} y$$

Let $\sigma = (q_\lambda \mid \lambda < \kappa)$ be a sequence of conditions in $\mathbb{Q}^*_\eta$. Let $\mathcal{F}$ be the weak compactness filter on κ (i.e., $\mathcal{F}$ is the filter generated by the sets $\{\alpha < \kappa \mid (V_\alpha, \in, A \cap V_\alpha) \models \phi\}$, for $A \subseteq V_\kappa$ and for a Π^1_1 sentence ϕ over $(V_\kappa, \in, A)$). $\mathcal{F}$ is a proper normal filter on κ.

Given $X \in \mathcal{F}^+$, say that

$$(q^L_\lambda \mid \lambda \in X), (q^R_\lambda \mid \lambda \in X)$$

is a separating pair for $(q_\lambda \mid \lambda < \kappa)$ if for all $\lambda \in X$:

Let $\sigma = (q_\lambda \mid \lambda < \kappa)$ be a sequence of conditions in $\mathbb{Q}^*_\eta$. Let $\mathcal{F}$ be the weak compactness filter on κ (i.e., $\mathcal{F}$ is the filter generated by the sets $\{\alpha < \kappa \mid (V_\alpha, \in, A \cap V_\alpha) \models \phi\}$, for $A \subseteq V_\kappa$ and for a Π^1_1 sentence ϕ over $(V_\kappa, \in, A)$). $\mathcal{F}$ is a proper normal filter on κ.

Given $X \in \mathcal{F}^+$, say that

$$(q^L_\lambda \mid \lambda \in X), (q^R_\lambda \mid \lambda \in X)$$

is a separating pair for $(q_\lambda \mid \lambda < \kappa)$ if for all $\lambda \in X$:

(1) Both of q^L_λ and q^R_λ extend q_λ.
(2) $\mathrm{dom}(f_{q^L_\lambda}) = \mathrm{dom}(f_{q^R_\lambda})$
(3) For all nonzero $\alpha \in \mathrm{dom}(f_{q^L_\lambda}) \cap M_\lambda$ and all $x \in \mathrm{dom}(f_{q^L_\lambda}(\alpha)) \setminus (\lambda \times \omega_1)$ and $y \in \mathrm{dom}(f_{q^R_\lambda}(\alpha)) \setminus (\lambda \times \omega_1)$, x and y are separated at stage α below λ by $q^L_\lambda \restriction \alpha$ and $q^R_\lambda \restriction \alpha$ via some pair $\chi_0(x, y, \alpha, \lambda)$, $\chi_1(x, y, \alpha, \lambda)$.
(4) The following holds for all $\lambda' > \lambda$ in X.
   (a) $q^L_\lambda \restriction M_\lambda = q^R_{\lambda'} \restriction M_{\lambda'}$
   (b) $q^L_\lambda \in M_{\lambda'}$
(5) The following holds for all $\lambda' > \lambda$ in X, all nonzero $\alpha \in \mathrm{dom}(q^L_\lambda) \cap \mathrm{dom}(q^R_{\lambda'})$, and all $x \in \mathrm{dom}(f_{q^L_\lambda(\alpha)}) \setminus (\lambda \times \omega_1)$ and $y' \in \mathrm{dom}(f_{q^R_{\lambda'}(\alpha)}) \setminus (\lambda' \times \omega_1)$.
   (a) $\alpha \in M_\lambda$
   (b) There are $x' \in \mathrm{dom}(f_{q^L_{\lambda'}(\alpha)}) \setminus (\lambda' \times \omega_1)$ and $y \in \mathrm{dom}(f_{q^R_\lambda(\alpha)}) \setminus (\lambda \times \omega_1)$ such that

$$\chi_0(x, y, \alpha, \lambda) = \chi_0(x', y', \alpha, \lambda')$$

and

$$\chi_1(x, y, \alpha, \lambda) = \chi_1(x', y', \alpha, \lambda')$$

The following claim is easy.

Claim

Let $X \in \mathcal{S}$ and suppose $\sigma^L = (q^L_\lambda \mid \lambda \in X)$, $\sigma^R = (q^R_\lambda \mid \lambda \in X)$ is a separating pair for σ. Then for all $\lambda < \lambda'$ in X,

$$q^L_\lambda$$

and

$$q^R_{\lambda'}$$

are compatible conditions.

Hence, it suffices to prove that there is $\sigma^L = (q^L_\lambda \mid \lambda \in X)$, $\sigma^R = (q^R_\lambda \mid \lambda \in X)$, a separating pair for σ. But this follows essentially from a construction in ω steps such that

* at every step we separate some given sequence of pair of nodes x, y,

followed by a pressing–down argument using the normality of $\mathcal{F}$.

The relevant separation, at every step of the construction, is effected via a Π^1_1 reflection argument: There is a measure 1 set C in $\mathcal{F}$ of $\lambda < \kappa$ such that, for relevant α,

- $M_\lambda \cap \mathbb{Q}_\alpha \lessdot \mathbb{Q}_\alpha$ and
- $M_\lambda \cap \mathbb{Q}_\alpha$ forces, over V, that $\dot{T}_\alpha \cap M_\lambda$ has no λ–branches.

Using this idea one can find suitable conditions

$$q_\lambda^{LL} \leq q_\lambda^{L}$$

and

$$q_\lambda^{RR} \leq q_\lambda^{R}$$

such that

- $q_\lambda^{LL} \restriction M_\lambda = q_\lambda^{RR} \restriction M_\lambda$ and
- x and y are separated by $q_\lambda^{LL} \restriction \alpha$ and $q_\lambda^{RR} \restriction \alpha$ at stage α below λ

(if this were not possible, we would be able to find λ–branches through $\dot{T}_\alpha \cap M_\lambda$ in the $M_\lambda \cap \mathbb{Q}_\alpha$–extension, which is impossible). $\square$

An open question

Question (Shelah): Is it consistent to have GCH together with a regular $\kappa \geq \omega_1$ such that SATP(κ) + SATP(κ^+)?