Generalized descriptive set theory under I0

Luca Motto Ros

Department of Mathematics "G. Peano"
University of Turin, Italy
luca.mottoros@unito.it
https://sites.google.com/site/lucamottoros/

KNAW Academy Colloquium "Generalised Baire Spaces"
Amsterdam, 23–24.08.2018

Joint work with Vincenzo Dimonte

Classical descriptive set theory

According to Kechris' book, "descriptive set theory is the study of **definable sets** in **Polish** (i.e. separable completely metrizable) **spaces**", and of their **regularity properties**.

Kechris, *Classical descriptive set theory*, 1995

Polish spaces: separable completely metrizable spaces, e.g. the *Cantor space* ${}^{\omega}2$ and the *Baire space* ${}^{\omega}\omega$.

Definable subsets: *Borel* sets, *analytic* sets, *projective* sets...

Regularity properties: Perfect set property (PSP), Baire property, Lebesgue measurability, ...

There has been various attempts to generalize classical DST to different setups, usually first varying the spaces under consideration, and then naturally adapting (some of) the other definitions to the new context.

Non-separable spaces

Drop separability from the definition of a Polish spaces (while keeping complete metrizability). Approach mainly motivated by analysis, where one deals with non-separable Banach spaces as well, and general topology.

A. H. Stone, *Non-separable Borel sets*, 1962

Baire spaces: $\prod_{n \in \omega} T_n$ where each T_n is discrete. In particular, the space $B(\lambda) = {}^{\omega}\lambda$ and, if $\mathrm{cof}(\lambda) = \omega$, the space $C(\lambda) = \prod_{i \in \omega} \lambda_i$, where the λ_i's are increasing and cofinal in λ (in symbols, $\lambda_i \nearrow \lambda$).

Definable sets: usual *Borel* sets (σ-algebra generated by open sets); λ-*analytic* sets = continuous images of $B(\lambda)$ (plus $\emptyset$).

Regularity properties: λ-PSP for a set A = either $|A| \leq \lambda$, or $B(\lambda)$ topologically embeds into A.

Among many other things, Stone showed e.g. that $B(\lambda) \approx C(\lambda)$ when $\lambda > \omega$, and that all Borel/λ-analytic subsets of $B(\lambda)$ have the λ-PSP.

Quasi-Polish spaces and alike

Drop (complete) metrizability, while keeping separability (or second-countability). Useful to encompass the study of topological spaces relevant to theoretical computer science which are not metrizable, like the ω-continuous domains (e.g. the *Scott domain* $\mathcal{P}(\omega)$).

M. de Brecht, *Quasi-Polish spaces*, 2013

Quasi-Polish spaces: separable spaces which are completely *quasi-metrizable*, where a quasi-metric is like a metric without the condition $d(x, y) = d(y, x)$.

Definable sets: usual *Borel* and *analytic/projective* sets.

Regularity properties: the usual ones, e.g. PSP, Baire property, and so on.

A general theory for such spaces can be fully developed: in fact, it turns out that quasi-Polish spaces are "almost" Polish, and differences occurs only at *finite* levels in the Borel hierarchy.

Generalized descriptive set theory

Don't care about separability and (complete) metrizability, but rather systematically replace ω with an uncountable cardinal κ in all definitions. Motivation not totally clear to me, but, *a posteriori*, remarkable connections with other areas of set theory and model theory (Shelah's stability theory).

(Too many authors to be cited, most of which are in this room...)

Generalized Cantor space and Baire space: ${}^{\kappa}2$ and ${}^{\kappa}\kappa$, endowed with the *bounded topology*, i.e. the topology generated by the sets $\boldsymbol{N}_s = \{x \in {}^{\kappa}2 \mid s \sqsubseteq x\}$ with $s \in {}^{<\kappa}2$ (and similarly for ${}^{\kappa}\kappa$).

Definable sets: κ^+-*Borel* sets = sets in the κ^+-algebra generated by open sets; κ-*analytic* sets = continuous images of closed subsets of ${}^{\kappa}\kappa$ (equivalently, continuous images of κ^+-Borel subsets of ${}^{\kappa}2$).

Regularity properties: κ-PSP for a set A = either $|A| \leq \kappa$, or ${}^{\kappa}2$ topologically embeds into A; κ-Baire property (when it makes sense); other "combinatorial" regularity properties.

Generalized descriptive set theory

Usually, generalized descriptive set theory is developed under the crucial condition

$$\kappa^{<\kappa} = \kappa \tag{†}$$

to ensure that e.g. both ${}^{\kappa}2$ and ${}^{\kappa}\kappa$ have a separability-like condition (i.e. they have a dense subset of size κ). This condition can equivalently be rewritten as

$$\kappa \text{ is regular} \qquad \text{and} \qquad 2^{<\kappa} = \kappa.$$

The (first half of the) assumption above causes the loss of metrizability when $\kappa > \omega$: indeed, ${}^{\kappa}2$ is (completely) metrizable iff ${}^{\kappa}2$ is first-countable iff $\mathrm{cof}(\kappa) = \omega$. (The same holds for ${}^{\kappa}\kappa$.)

The resulting theory is extremely rich and interesting, but quite different from the classical one: most of the nontrivial results are either simply false or at least independent of ZFC when $\kappa > \omega$ (e.g. both the *Lusin's separation theorem* and *Souslin's theorem* fail).

Generalized descriptive set theory

A general trend emerging from various papers is

> Large cardinals (expecially when κ itself is a large cardinal) allow to preserve a bit more of the classical picture.

For example, ${}^{\kappa}\kappa \not\approx {}^{\kappa}2$ if (and only if) κ is weakly compact. On the other extreme, the generalized Cantor and Baire spaces enjoy all possible "pathologies" in the constructible universe L.

Many years ago, Džamonja suggested that maybe singular cardinals could give a better picture. Indeed, together with Väänänen, she studied a bit of generalized descriptive set theory with κ singular, mainly in connection with model theory (chainable models).

More recently, Woodin suggested to study generalized DST under I0 in connection with his study of the model $\mathrm{L}(\mathrm{V}_{\lambda+1})$ (where λ is the witness of I0). Notice that such a λ has always countable cofinality.

I0 and Woodin's analysis

The axiom I0

I0(λ) is the statement: There is a nontrivial elementary embedding $j \colon \mathrm{L}(\mathrm{V}_{\lambda+1}) \prec \mathrm{L}(\mathrm{V}_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$ (we call j a *witness* to I0(λ)).

I0 is the statement: there is λ for which I0(λ).

Woodin considers $\mathrm{V}_{\lambda+2} = \mathcal{P}(\mathrm{V}_{\lambda+1})$ as a large cardinal version of $\mathcal{P}({}^{\omega}2)$: indeed, one can see V_λ as an analogue of $\mathrm{V}_\omega \approx \omega$, so that $V_{\lambda+1} = \mathcal{P}(\mathrm{V}_\lambda)$ is the analogue of $\mathcal{P}(\omega) \approx {}^{\omega}2$. Following this analogy, Woodin considers the topology on $\mathrm{V}_{\lambda+1}$ generated by the sets of the form

$$O_{a,\alpha} = \{X \in V_{\lambda+1} \mid X \cap \mathrm{V}_\alpha = a\}$$

for $\alpha < \lambda$ and $a \subseteq \mathrm{V}_\alpha$.

Woodin claims that "*the theory of* $\mathcal{P}(\mathrm{V}_{\lambda+1})$ *in* $\mathrm{L}(\mathrm{V}_{\lambda+1})$ *under* I0(λ) *is reminiscent of the theory of* $\mathcal{P}(\mathbb{R})$ *in* $\mathrm{L}(\mathbb{R}) = \mathrm{L}(\mathrm{V}_{\omega+1})$ *under* AD".

I0 and Woodin's analysis

A test for Woodin's claim is the Perfect Set Property PSP. Some of the following statements involve $U(j)$-**representability**, which is a technical notion isolated by Woodin reminiscent of the one of κ-*weakly homogenously Souslin* sets.

Theorem (Woodin)

Assume I0(λ), as witnessed by j. Every $U(j)$-representable set $A \subseteq \mathrm{V}_{\lambda+1}$ in $\mathrm{L}(\mathrm{V}_{\lambda+1})$ satisfies the following dichotomy: either $|A| \leq \lambda$ or ${}^{\omega}2$ topologically embeds into A.

Theorem (Shi)

Assume I0(λ), as witnessed by j. Then every set A in $\mathrm{L}_{\lambda}(\mathrm{V}_{\lambda+1})$ satisfies the following dichotomy: either $|A| \leq \lambda$ or $C(\lambda) = \prod_{i \in \omega} \lambda_i$ topologically embeds into A, where $\lambda_i \nearrow \lambda$.

I0 and Woodin's analysis

Theorem (Shi)

Assume I0(λ), as witnessed by j. Assume that all subsets of $\mathrm{V}_{\lambda+1}$ in $\mathrm{L}(\mathrm{V}_{\lambda+1})$ are $U(j)$-representable. Then every $A \subseteq \mathrm{V}_{\lambda+1}$ in $\mathrm{L}(\mathrm{V}_{\lambda+1})$ satisfies the following dichotomy: either $|A| \leq \lambda$ or $C(\lambda) = \prod_{i\in\omega} \lambda_i$ topologically embeds into A, where $\lambda_i \nearrow \lambda$.

Theorem (Cramer)

Assume I0(λ), as witnessed by j. Every $A \subseteq \mathrm{V}_{\lambda+1}$ in $\mathrm{L}(\mathrm{V}_{\lambda+1})$ satisfies the following dichotomy: either $|A| \leq \lambda$ or $B(\lambda) = {}^{\omega}\lambda$ topologically embeds into A.

The proofs are remarkably long and complicated, heavily using forcing, absoluteness, and a great part of Woodin's machinery.

Back to generalized DST: the singular case

Our goal is to study the generalized Cantor space ${}^{\lambda}2$ when λ is singular. We denote by λ_i a(ny) sequence of length $\mu = \mathrm{cof}(\lambda)$ cofinal in λ.

Proposition (Džamonja-Väänänen, Dimonte-M.)

The following spaces are homeomorphic (products of length μ are endowed with the $< \mu$-supported product topology):

- ${}^{\lambda}2$;
- $\prod_{i<\mu} 2^{\lambda_i}$, where each 2^{λ_i} is discrete;
- ${}^{\mu}(2^{<\lambda})$, where $2^{<\lambda}$ is discrete.

The generalized Cantor space ${}^{\lambda}2$

Dropping the first half of the usual condition

$$\lambda^{<\lambda} = \lambda \qquad \equiv \qquad \operatorname{cof}(\lambda) = \lambda \text{ and } 2^{<\lambda} = \lambda \tag{$\dagger$}$$

we remain with a singular λ satisfying $2^{<\lambda} = \lambda$ or, equivalently, with a singular *strong limit* λ. In this situation, ${}^{\lambda}2$ still has density λ and the previous result reads as

$${}^{\lambda}2 \approx \prod_{i<\mu} \lambda_i \approx {}^{\mu}\lambda.$$

Moreover, in this case ${}^{\lambda}2 \not\approx {}^{\lambda}\lambda$ because the latter has density $\lambda^{<\lambda} > \lambda$.
(Indeed, ${}^{\lambda}2$ and ${}^{\lambda}\lambda$ may even fail to be (λ^+-)Borel isomorphic.)

If furthermore $\operatorname{cof}(\lambda) = \omega$, then we get

$${}^{\lambda}2 \approx C(\lambda) \approx B(\lambda).$$

Thus when λ is strong limit of countable cofinality, the generalized Cantor space ${}^{\lambda}2$ is a completely metrizable space of density λ, briefly: a λ-**Polish space**. These very simple observations have lot consequences.

The generalized Cantor space ${}^{\lambda}2$

A space X is said **uniformly zero-dimensional** if for every $\varepsilon > 0$, every open set of X can be partitioned into *clopen* sets with diameter $< \varepsilon$.

(Uniform zero-dimensionality follows from ultrametrizability and is equivalent to ultraparacompactness.)

Proposition (Dimonte-M.)

Let $\lambda > \omega$ be strong limit of countable cofinality.

- ${}^{\lambda}2$ is universal for uniformly zero-dimensional λ-Polish spaces, that is: X is λ-Polish and uniformly zero-dimensional iff it is homeomorphic to a closed subset of ${}^{\lambda}2$, iff it admits a compatible complete ultrametric.
- In a uniformly zero-dimensional λ-Polish space X, every closed set $C \subseteq X$ is a retract, i.e. there is a continuous surjection $g \colon X \to C$ with $g \restriction C = \mathrm{id}_C$.
- Every nonempty λ-Polish space is a continuous image of ${}^{\lambda}2$.

The generalized Cantor space ${}^{\lambda}2$ and Woodin's $\mathrm{L}(\mathrm{V}_{\lambda+1})$

Woodin's approach to the study of $\mathrm{V}_{\lambda+1}$ falls in this setup as well. Recall that $\mathrm{V}_{\lambda+1}$ is endowed with the topology generated by $O_{a,\alpha} = \{X \in V_{\lambda+1} \mid X \cap \mathrm{V}_\alpha = a\}$ for $\alpha < \lambda$ and $a \subseteq \mathrm{V}_\alpha$.

Lemma

If $\mathrm{cof}(\lambda) = \omega$ and $\lambda_i \nearrow \lambda$, then

$$\mathrm{V}_{\lambda+1} \approx \prod_{i\in\omega} |\mathrm{V}_{\lambda_i+1}| \approx {}^{\omega}\Big(\sup_{i\in\omega} \beth_{\lambda_i+1}\Big) \approx {}^{\omega}\Big(\beth_\lambda\Big).$$

If furthermore λ is limit of inaccessible cardinals (which is the case under I0(λ)), then

$$\mathrm{V}_{\lambda+1} \approx \prod_{i\in\omega} \lambda_i \approx {}^{\omega}\lambda \approx {}^{\lambda}2.$$

λ-Borel sets

As usual, on ${}^{\lambda}2$ we consider λ^+-**Borel** sets. It can be proven that these sets can be stratified in a hierarchy with exactly λ^+-many levels (if $2^{<\lambda} > \lambda$, a new proof is needed for the non-collapsing part).

Notice also that if λ is singular then

$$\lambda^+\text{-Borel} = \lambda\text{-Borel}.$$

Similar results hold for the generalized Baire space ${}^{\lambda}\lambda$.

λ-Analytic sets

In the classical case, the following conditions (defining **analytic** sets) are equivalent:

1. A is a continuous image of a Polish space
2. $A = \emptyset$ or A is a continuous image of ${}^{\omega}\omega$
3. A is a continuous image of a closed $F \subseteq {}^{\omega}\omega$
4. A is a continuous/Borel image of a Borel subset of ${}^{\omega}2$
5. A is the projection of a closed subset of $X \times {}^{\omega}\omega$
6. A is the projection of a Borel subset of $X \times {}^{\omega}2$.

There are some problems when trying to generalize these equivalences by replacing ${}^{\omega}2$ and ${}^{\omega}\omega$ with ${}^{\kappa}2$ and ${}^{\kappa}\kappa$, especially when κ is regular.

However...

λ-Analytic sets

If $\operatorname{cof}(\lambda) = \omega$ and λ is strong limit, TFAE:

1. A is a continuous image of a λ-Polish space
2. $A = \emptyset$ or A is a continuous image of ${}^{\omega}\lambda$
3. A is a continuous image of a closed $F \subseteq {}^{\omega}\lambda$
4. A is a continuous/Borel image of a Borel subset of ${}^{\lambda}2$
5. A is the projection of a closed subset of $X \times {}^{\omega}\lambda$
6. A is the projection of a Borel subset of $X \times {}^{\lambda}2$.

This is exactly the notion of a λ-**analytic** set isolated by Stone.

Remark: One may be tempted to generalize the notion of "analytic" as "continuous image of a closed subset of ${}^{\lambda}\lambda$", as in the regular case. However, this would give a much coarser definition, encompassing λ-analytic sets, λ-coanalytic sets, $\mathbf{\Sigma}^1_2(\lambda)$ sets, and, under the assumption that $\lambda^{<\lambda}$ is large, also all λ-projective sets.

λ-analytic vs λ-Borel

Assume again that λ is strong limit with countable cofinality.

Proposition (Dimonte-M.)

The collection of all λ-analytic sets (properly) contain the $\lambda^{(+)}$-Borel ones.

Generalized Lusin's separation theorem (Dimonte-M.)

If A, B are disjoint analytic subsets of a λ-Polish space, then A can be separated from B by a λ-Borel set.

Generalized Souslin's theorem (Dimonte-M.)

A subsets of a λ-Polish space is λ-bianalytic iff it is $\lambda^{(+)}$-Borel.

This has many consequences:

- a function is λ-Borel iff its graph is λ-analytic, iff its graph is λ-Borel;
- the injective λ-Borel image of a λ-Borel set is still λ-Borel;
- ...

λ-Perfect set property

Definition

A subset A of a topological space X has the λ-PSP if either $|A| \leq \lambda$, or else ${}^{\lambda}2$ topologically embeds into A.

Similarly to the classical case

Theorem (essentially A. H. Stone)

Let λ be strong limit of countable cofinality. Every λ-analytic subset of a uniformly zero-dimensional λ-Polish space has the λ-PSP.

What for more complicated sets?

Motivated by the fact that, in the classical context, κ-homogeneously Souslin sets have the PSP (and inspired by Woodin's notion of $U(j)$-representability), we developed the following machinery.

$(\mathbb{U}, \kappa)$-representable sets

Definition

A family $\mathbb{U}$ of ultrafilters is **orderly** iff there exists a set K such that for all $\mathcal{U} \in \mathbb{U}$ there is $n \in \omega$ for which ${}^n K \in \mathcal{U}$. Such an n is called the **level** of $\mathcal{U}$.

A **tower** of ultrafilters in such a $\mathbb{U}$ is a sequence $(\mathcal{U}_i)_{i \in \omega}$ such that for all $m < n < \omega$:

- $\mathcal{U}_n \in \mathbb{U}$ has level n;
- $\mathcal{U}_n$ *projects* to $\mathcal{U}_m$, i.e. for each $A \subseteq {}^m K$ we have

$$A \in \mathcal{U}_m \iff \{s \in {}^n K \mid s \restriction m \in A\} \in \mathcal{U}_n.$$

A tower of ultrafilters $(\mathcal{U}_i)_{i \in \omega}$ is **well-founded** iff for every sequence $(A_i)_{i \in \omega}$ with $A_i \in \mathcal{U}_i$ there is $z \in {}^\omega K$ such that $z \restriction i \in A_i$ for all $i \in \omega$.

$(\mathbb{U}, \kappa)$-representable sets

From now on λ is strong limit with $\mathrm{cof}(\lambda) = \omega$, and $\lambda_i \nearrow \lambda$.

Definition

Let $\kappa \geq \lambda$ be a cardinal, and let $\mathbb{U}$ be an orderly family of *κ-complete* ultrafilters. A $(\mathbb{U}, \kappa)$-**representation** for $Z \subseteq {}^{\omega}\lambda$ is a function $\pi \colon \bigcup_{i \in \omega} {}^{i}\lambda \times {}^{i}\lambda \to \mathbb{U}$ such that:

- if $s, t \in {}^{i}\lambda$, then $\pi(s,t)$ has level i;
- for any $(s,t) \in {}^{n}\lambda$ if $(s', t') \sqsupseteq (s,t)$ then $\pi(s', t')$ projects to $\pi(s,t)$;
- $x \in Z$ iff there is $y \in {}^{\omega}\lambda$ s.t. $(\pi(x \restriction i, y \restriction i))_{i \in \omega}$ is well-founded.

Remark 1: If $\lambda = \omega$ and $A \subseteq {}^{\omega}\omega$ is κ-weakly homogenously Souslin, then A is $(\mathbb{U}, \kappa)$-representable for a suitable orderly family of ultrafilters $\mathbb{U}$.

Remark 2: Exploiting the natural homeomorphism between $\mathrm{V}_{\lambda+1}$ and ${}^{\omega}\lambda$ the above definition yields Woodin's $U(j)$-representability when $\kappa = \lambda^{+}$ and $\mathbb{U}$ is a certain family of ultrafilters usually denoted by $\mathbb{U}(j, \kappa, (a_i)_{i \in \omega})$.

Tower condition

The following condition turns out to be very helpful when checking well-foundness of towers of ultrafilters.

Definition

A $(\mathbb{U}, \kappa)$-representation π for a set $Z \subseteq {}^{\omega}\lambda$ has the **tower condition** if there exists $F \colon \operatorname{ran} \pi \to \bigcup \mathbb{U}$ such that:

- $F(\mathcal{U}) \in \mathcal{U}$ for all $\mathcal{U} \in \operatorname{ran}(\pi)$;
- for every $x, y \in {}^{\omega}\lambda$, the tower of ultrafilters $(\pi(x \restriction i, y \restriction i))_{i \in \omega}$ is well-founded iff there is $z \in {}^{\omega}K$ such that $z \restriction i \in F(\pi(x \restriction i, y \restriction i))$ for all $i \in \omega$.

Remark 1: When $\lambda = \omega$, the tower condition automatically follows from the κ-weakly homogeneously Souslin condition.

Remark 2: Woodin has an analogous notion of "tower condition" in the context of $U(j)$-representability. Cramer later proved that if I0(λ) holds, then all $U(j)$-representable sets in $\mathcal{P}(\mathrm{V}_{\lambda+1}) \cap \mathrm{L}(\mathrm{V}_{\lambda+1})$ admit in fact a $U(j)$-representation with the tower condition.

Main theorem

Here is our main theorem in this direction.

Theorem (Dimonte-M.)

Let λ be strong limit with $\mathrm{cof}(\lambda) = \omega$, and let $\kappa \geq \lambda$ be a cardinal. If $Z \subseteq {}^{\omega}\lambda$ admits a $(\mathbb{U}, \kappa)$-representation with the tower condition, then Z has the λ-PSP.

The proof of this result uses only elementary arguments and exploit some variants of classical games: no forcing/absoluteness/Woodin's machinery is involved...

Perfect set property under I0

Corollary

Assume I0(λ), as witnessed by j. If $A \in \mathcal{P}(\mathrm{V}_{\lambda+1}) \cap \mathrm{L}(\mathrm{V}_{\lambda+1})$ is $U(j)$-representable, then A has the λ-PSP.

Corollary

Assume I0(λ). All λ-projective subsets of any uniformly zero-dimensional λ-Polish space have the λ-PSP.

Corollary (of the proof of the main theorem)

Assume I0(λ), as witnessed by a proper j with $\mathrm{crt}(j) = \kappa$. Let $\mathbb{P}$ be the Prikry forcing on κ with respect to the measure generated by j. Then there exists a $\mathbb{P}$-generic extension $\mathrm{V}[G]$ of V in which all κ-projective subsets of any uniformly zero-dimensional κ-Polish space have the κ-PSP.

Remark: The cardinal κ is much smaller than λ, and possibly does not satisfy I0 in the generic extension.

Skip

Proof of the main theorem

Theorem (Dimonte-M.)

Let λ be strong limit with $\mathrm{cof}(\lambda) = \omega$, and let $\kappa \geq \lambda$ be a cardinal. If $Z \subseteq {}^{\omega}\lambda$ admits a $(\mathbb{U}, \kappa)$-representation with the tower condition, then Z has the λ-PSP.

Proof. Let π be a $(\mathbb{U}, \kappa)$-representation for Z with the tower condition, as witnessed by F. Let $G(Z)$ (or rather $G(\pi, F)$) be the game

I	$\langle j_0, (s_i^0, t_i^0)_{i<\lambda_0} \rangle$	$z_0, \langle j_1, (s_i^1, t_i^1)_{i<\lambda_1} \rangle$	$z_1, \langle j_2, (s_i^2, t_i^2)_{i<\lambda_2} \rangle$	$\dots$
II	i_0	i_1	i_2	$\dots$

- $j_k \in \omega$, $s_i^k, t_i^k \in {}^{j_k}\mu_k$ for some $\mu_k < \lambda$, and $s_i^k \neq s_{i'}^k$ if $i \neq i'$;
- $i_k < \lambda_k$;
- $z_k \in F(\pi(s_{i_k}^k, t_{i_k}^k))$;
- $j_{k+1} > j_k$, $s_i^{k+1} \sqsupseteq s_{i_k}^k$ and $t_i^{k+1} \sqsupseteq t_{i_k}^k$ for all $i < \lambda_{k+1}$, and $z_{k+1} \sqsupseteq z_k$.

I wins if she can play for infinitely many turns.

Proof of the main theorem

When I wins a run, she has built an element $x = \bigcup_{k \in \omega} s^k_{i_k} \in {}^\omega\lambda$, and a $y = \bigcup_{k \in \omega} t^k_{i_k} \in {}^\omega\lambda$ witnessing that $x \in Z$ — the well-foundedness of the corresponding tower is witnessed by $z = \bigcup_{k \in \omega} z_k$, since $z_k \in F(\pi(s^k_{i_k}, t^k_{i_k}))$.

$G(Z)$ is a closed game, hence determined. If I has a winning strategy, testing it against all possible moves of II we get an embedding of $\prod_{k \in \omega} \lambda_k = C(\lambda) \approx {}^\lambda 2$ into Z. So let us assume that II has a winning strategy τ in $G(Z)$.

Consider the ausiliary game $G^*(Z)$ (or rather $G(\pi)$)

I	$\langle j_0, (s^0_i, t^0_i)_{i<\lambda_0} \rangle$	$\langle j_1, (s^1_i, t^1_i)_{i<\lambda_1} \rangle$	$\langle j_2, (s^2_i, t^2_i)_{i<\lambda_2} \rangle$	$\dots$
II	i_0	i_1	i_2	$\dots$

where I *does not have to produce the witnesses* z_k, and I wins iff $x = \bigcup_{k \in \omega} s^k_{i_k} \in Z$ with $y = t^k_{i_k}$ witnessing this. Such a game is not necessarily determined (the complexity of the payoff depends on the complexity of Z and π), but...

Proof of the main theorem

...any winning strategy τ of II in $G(Z)$ can be converted into a winning strategy τ^* of II in $G^*(Z)$.

The idea is that II simulates a run in $G(Z)$ testing all possible $z_k \in F(\pi(s^k_{i_k}, t^k_{i_k}))$ that I could play. Using λ-completeness of $\pi(s^k_{i_k}, t^k_{i_k})$, for a measure-one set of these possibilities τ will suggest the very same move $\bar{\imath}_k$: then II plays precisely this $\bar{\imath}_k$ in his corresponding turn in $G^*(Z)$.

Claim. If II wins $G^*(Z)$, then $|Z| \leq \lambda$.

Given a position p in the game $G^*(Z)$ consisting of k-many rounds, let A_p be the set of those $s^{k-1}_{i_{k-1}} \sqsubseteq x \in {}^\omega\lambda$ for which whatever I plays in her next turn, the answer by II following τ^* is such that $s^k_{i_k} \not\sqsubseteq x$. Arguing as in the classical case, one gets $|A_p| \leq (\lambda_k)^\omega < \lambda$. Moreover, $Z \subseteq \bigcup_p A_p$ because any $x \in Z \setminus \bigcup_p A_p$ would yield a strategy for I in $G^*(Z)$ defeating τ^*. Finally, a direct computation shows that there are only λ-many possible positions p in $G^*(Z)$, whence $|Z| \leq \left|\bigcup_p A_p\right| \leq \lambda \cdot \lambda = \lambda$. □

The end

Thank you for your attention!