Axioms of determinacy and their set-theoretic strength

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Outline

Determinacy Axioms

Set-theoretic Strength

Questions for Ph.D topic

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Infinite games

1. Infinite games with perfect information
   - Axiom of regular (Gale-Stewart) determinacy (AD): For any $A \subseteq {}^{\omega}\omega$, $G_{\omega}(A)$ is determined.
2. Infinite games with imperfect information
   - Axiom of Blackwell determinacy (Bl-AD): For any $A \subseteq {}^{\omega}\omega$, $B_{\omega}(A)$ is determined.

Remark

- AD implies Bl-AD.
- The converse is unknown (Martin's Conjecture).
- Con(AD) $\iff$ Con(Bl-AD).
- We can prove some consequences of AD from Bl-AD.

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Determinacy vs Axiom of Choice

- AD, Bl-AD are inconsistent with AC.
- AD, Bl-AD imply many interesting statements contradicting with AC.
- Many restricted versions of AD, Bl-AD are consistent with AC (e.g. Projective determinacy).

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Consistency strength

- Many mathematical questions are undetermined in ZF or ZFC.
- We need additional axioms to resolve them.
- How do we compare them? $\Rightarrow$ via "consistency strength"

S, T : theories

- If Con(S) $\Rightarrow$ Con(T), then S $\geq$ T.
- If Con(S) $\Rightarrow$ Con(T) and we cannot derive Con(T) $\Rightarrow$ Con(S), then S $>$ T.

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What are Large Cardinals?

- Uncountable cardinals.
- Generalizations of ω: some transcendental properties for smaller cardinals.

Example

- Inaccessible cardinals:
  - κ is *inaccessible* if κ is regular and $(\forall\lambda < \kappa)\ 2^\lambda < \kappa$.
- Weakly compact cardinals:
  - κ is *weakly compact* if the compactness theorem holds for $\mathcal{L}_{\kappa,\kappa}$ in a weak sense.
- Strongly compact cardinals:
  - κ is *strongly compact* if the compactness theorem holds for $\mathcal{L}_{\kappa,\kappa}$ in a strong sense.
- Measurable cardinals:
  - κ is *measurable* if there is a non-principal κ-complete ultrafilter on κ.

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What is good in Large Cardinals?

1. Large enough: we can resolve many mathematical questions undetermined in ZFC.
2. Almost all large cardinals are linearly ordered via consistency strength.
3. Many mathematical statements are consistent or equiconsistent with some large cardinals.
   $\Rightarrow$Large cardinals are a standard measure via consistency strength.

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Determinacy and Large Cardinals

Theorem (Woodin et al.)

1. The following are equiconsistent:
   - ZF $+$ AD.
   - ZFC $+$ “There are infinitary many Woodin cardinals.”
2. The following are equiconsistent:
   - ZFC $+$ Δ^1_2-determinacy.
   - ZFC $+$ “There is a Woodin cardinal.”
3. The following are logically equivalent.
   - ZFC $+$ Π^1_1-determinacy.
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Interesting Questions

1. Develop Blackwell Determinacy Theory.
   - Does Bl-AD imply that every set of reals has the Baire property?
   - Does Bl-AD imply Moschovakis Coding Lemma?
2. Consistency strength of Higher Blackwell Determinacy.
   - Consistency strength of Bl-$AD_{\mathbb{R}}$.
     cf. We only know that Con(Bl-$AD_{\mathbb{R}}$) $>$ Con(AD).
   - Consistency strength of Bl-$AD_{\mathbb{R}}$ $+$ "Θ is regular".
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