Will my allocation be conflict-prone ?

A scale of properties for characterizing resource allocation instances

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STeamer - LIG

Spatio-temporal information, adaptability, multimedia and knowledge representation

Fair division of indivisible goods...

We have:

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- an allocation $\overrightarrow{\pi} : \mathcal{A} \to 2^{\mathcal{O}}$
- such that $\pi_i \cap \pi_j = \emptyset$ if $i \neq j$ (preemption),
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Plenty of real-world applications: course allocation, operation of Earth observing satellites, ...

Centralized allocation

A classical way to solve the problem:

- Ask each agent i to give a score (weight, utility...) $w_i(o)$ to each object o
- Consider all the agents have **additive** preferences
  $\rightarrow u_i(\pi) = \sum_{o \in \pi} w_i(o)$
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- Find an allocation $\overrightarrow{\pi}$ that:

1. maximizes the collective utility defined by a **collective utility function**,
   e.g. $uc(\overrightarrow{\pi}) = \min_{i \in \mathcal{A}} u(\pi_i)$ – egalitarian solution
   [Bansal and Sviridenko, 2006]
2. or satisfies a given **fairness criterion**,
   e.g. $u_i(\pi_i) \geq u_i(\pi_j)$ for all agents i, j – envy-freeness
   [Lipton et al., 2004].

Bansal, N. and Sviridenko, M. (2006).
The Santa Claus problem.
In *Proceedings of STOC'06*. ACM.

Lipton, R., Markakis, E., Mossel, E., and Saberi, A. (2004).
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$\overrightarrow{\pi}$ is **not** envy-free (agent 1 envies agent 2)

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Fairness properties

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Idea: consider several fairness properties, and try to satisfy the most demanding one.
In this work we consider five such properties.

Outline

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- Deciding whether an allocation is envy-free is easy (quadratic time).
- Deciding whether an instance (agents, objects, preferences) has an envy-free allocation is hard – **NP**-complete [Lipton et al., 2004].

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Proportional fair share

Proportional fair share (PFS):

- Initially defined by Steinhaus [Steinhaus, 1948] for continuous fair division (*cake-cutting*)
- **Idea:** each agent is "entitled" to at least the nth of the entire resource

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The **proportional fair share** of an agent i is equal to:

$$u_i^{\mathrm{PFS}} \stackrel{def}{=} \frac{u_i(\mathcal{O})}{n} = \sum_{o \in \mathcal{O}} \frac{w_i(o)}{n}$$

An allocation $\overrightarrow{\pi}$ satisfies **(proportional) fair share** if every agent gets at least her fair share.

Proportional fair share: facts

Easy or known facts:

- Deciding whether an allocation satisfies proportional fair share (PFS) is easy (linear time).
- For a given instance, there may be no allocation satisfying PFS
  $\rightarrow$ *e.g.* 2 agents, 1 object
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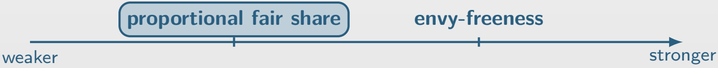

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- **Idea:** in the **cake-cutting** case, PFS = the best share an agent can hopefully get for sure in a *"I cut, you choose (I choose last)"* game.
- Same game for indivisible goods $\rightarrow$ MFS.

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An allocation $\vec{\pi}$ satisfies **max-min fair share** (MFS) if every agent gets at least her max-min fair share.

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Example: 2 agents, 1 object.

$u_1^{\mathrm{MFS}} = u_2^{\mathrm{MFS}} = 0 \rightarrow$ every allocation satisfies MFS!

Not very satisfactory, but can we do much better?

Max-min fair share: properties

Facts:

- Computing u_i^{MFS} for a given agent is hard $\rightarrow$ **NP**-complete [Partition]
- Hence, deciding whether an allocation satisfies MFS is also hard.
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Intuition:

- the situation where all agents have the same preferences is the **worst** possible situation
- in that situation, an allocation satisfying MFS exists (see definition)
- all other situation makes every agent better off.

Max-min fair share: special cases

Special cases: conjecture proved for:

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- mFS $=$ the worst share an agent can get in a *"Someone cuts, I choose first"* game.
- In the **cake-cutting** case, same as PFS.

Min-max fair share: properties

Facts:

- Computing u_i^{mFS} for a given agent is hard $\rightarrow$ **coNP**-complete [PARTITION]
- Hence, deciding whether an allocation satisfies mFS is also hard.
- $\overrightarrow{\pi}$ satisfies mFS $\Rightarrow$ $\overrightarrow{\pi}$ satisfies PFS.
- $\overrightarrow{\pi}$ is envy-free $\Rightarrow$ $\overrightarrow{\pi}$ satisfies mFS.

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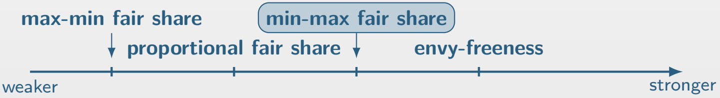

Competitive Equilibrium from Equal Incomes (CEEI)

- Set one price $p_o \leq$ £1 for each object o.
- Give £1 to each agent i.
- Let $\pi_i^\star$ be (among) the best share(s) agent i can buy with her £1.
- If $(\pi_1^\star, \ldots, \pi_n^\star)$ is a valid allocation, it forms, together with $\overrightarrow{p}$, a **CEEI**.

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Allocation $\overrightarrow{\pi}$ satisfies CEEI if $\exists \overrightarrow{p}$ such that $(\overrightarrow{\pi}, \overrightarrow{p})$ is a CEEI.

- Classical notion in economics [Moulin, 1995]
- Not so much studied in computer science – [Othman et al., 2010] is an exception

Moulin, H. (1995).
Cooperative Microeconomics, A Game-Theoretic Introduction.
Prentice Hall.

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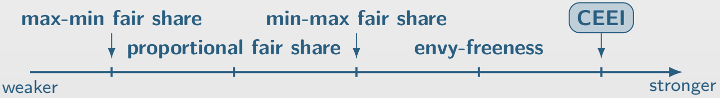
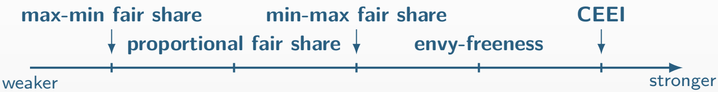


Interpretation



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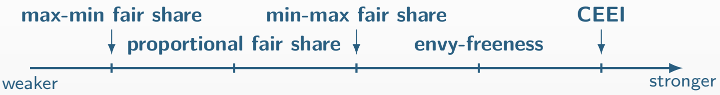

1. For all allocation $\overrightarrow{\pi}$:

$$(\overrightarrow{\pi} \models \text{CEEI}) \Rightarrow (\overrightarrow{\pi} \models \text{EF}) \Rightarrow (\overrightarrow{\pi} \models \text{mFS}) \Rightarrow (\overrightarrow{\pi} \models \text{PFS}) \Rightarrow (\overrightarrow{\pi} \models \text{MFS})$$

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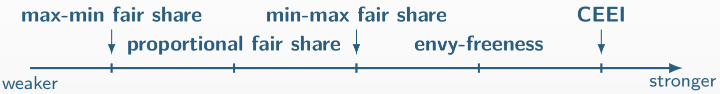

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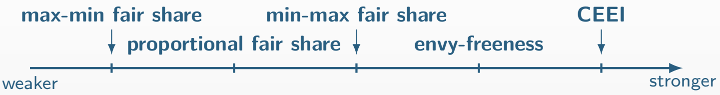

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Two extreme examples:

- 2 agents, 1 object $\rightarrow$ only in $\mathcal{I}_{|\text{MFS}}$
- 2 agents, 2 objects, with

	1	2
agent 1	1000	0
agent 2	0	1000

$\rightarrow$ in $\mathcal{I}_{|\text{CEEI}}$ (with *e.g.* $\overrightarrow{p} = \langle 1, 1 \rangle$).

Outline

Strict inclusions?

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Are these inclusions strict?

- From MFS to PFS: two agents, one object.
- From PFS to mFS: an example with 3 agents, 3 objects found.
- From mFS to EF: not straightforward, but one example with 3 agents, 4 objects found.
- From EF to CEEI: no example found[1], but very likely to be strict by computational complexity arguments.

[1] because it seems algorithmically hard to compute a CEEI...

Back to egalitarianism...

Other approach to fairness... Find an allocation $\overrightarrow{\pi}$ that:

1. maximizes the collective utility defined by a **collective utility function**, *e.g.* $uc(\overrightarrow{\pi}) = \min_{i \in \mathcal{A}} u(\pi_i)$ – egalitarian solution

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- **Envy-freeness:** question studied in [Brams and King, 2005]
- **Max-min fair share:** egalitarian optimal allocations **almost always satisfy** max-min fair share.

	1	2	3	4	
agent 1	58	†15	†*19	8	→ *19 / †34
agent 2	†63	*5	25	*7	→ *12 / †63
agent 3	37	10	*27	†26	→ *27 / †26

3 agents, 4 objects: about 1 counterexample for 3500 instances

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Note:

- Egalitarianism requires the preferences to be **comparable**:
  - either expressed on a same scale (*e.g.* money)...
  - ...or normalized (*e.g.* Kalai-Smorodinsky)
- The five fairness criteria introduced do not (**independence of the individual utility scales**).

$\rightarrow$ This is a very appealing property.

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A weight $w(\mathcal{S})$ to each subset $\mathcal{S}$ of objects (not only singletons) of size $\leq k$.
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Examples:

- $w(skis) = 10$; $w(poles) = 0$; $w(\{skis, poles\}) = 90$
  $\rightarrow u(\{skis, poles\}) = 100 > 10 + 0$
- $w(skis) = 100$; $w(snowboard) = 100$; $w(\{skis, snowboard\}) = -100$
  $\rightarrow u(\{skis, snowboard\}) = 100 < 100 + 100$

MFS and k-additive preferences

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Conjecture

For each instance there is at least one allocation that satisfies max-min fair share.

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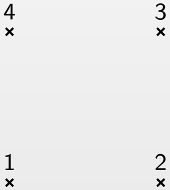
For k-additive preferences ($k \geq 2$) this is obviously not true:

Example: 4 objects, 2 agents



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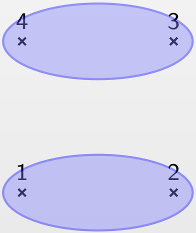

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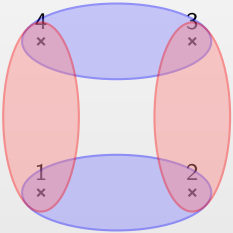

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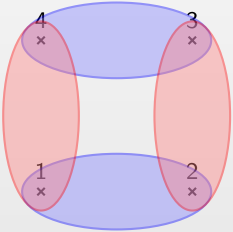

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Worse... Deciding whether there exists one is **NP**-complete [Partition].

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A possible approach to fairness in multiagent resource allocation problems:

1. Determine the highest satisfiable criterion.
2. Find an allocation that satisfies this criterion.
3. Explain to the upset agents that we cannot do much better.

Future directions

- Close the **conjecture** and missing complexity results.
- Develop efficient **algorithms** (possibly in conjunction with approximation of fairness criteria)
- **Experiments**: Build a cartography of resource allocation problems.
- Extend the results to **more expressive preference languages**.

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- Extend the results to **more expressive preference languages**.

- The five criteria do not require interpersonal comparison of utilities.
- Moreover: Four of them are **purely ordinal** (PFS is not)
- Do the results extend to (separable) **ordinal preferences** ?