Real Candidacy Games

Itay Sabato, Svetlana Obraztsova, Zinovi Rabinovich & Jeffrey S. Rosenschein

In Short

- A new model that extends Strategic Candidacy Games.
- Candidates may choose either to quit or to join the election at a real position.
- Voter positions are fixed and their preferences are determined by the distances from the candidates.
- Best response strategies are poly-time computable for any polynomial voting rule.
- Results on existence of Pure Nash Equilibria for Condorcet-consistent voting rules and positional scoring rules.

Model and Notation

Basics

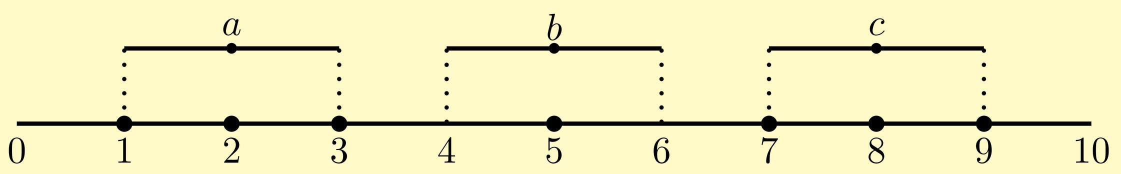
- The set of voters is $V = \{1, \dots, n\}$.
- The set of candidates is $C = \{c_1, \dots, c_m\}$.
- Voter positions are given as $p = (p_1, \dots, p_n) \in \mathbb{R}^n$.
- Each candidate c_i chooses a strategy $s_i = s_{c_i} \in \mathbb{R}_\perp = \mathbb{R} \cup \{\perp\}$ where $\perp$ denotes withdrawal of candidacy.
- The candidate position vector, AKA *state*, is $s = (s_1, \dots, s_m) \in \mathbb{R}^m_\perp$.
- The election winner is denoted by $\mathcal{V}(p, s)$ or simply $\mathcal{V}(s)$.

Preferences

- The positions p, s are mapped to a preference profile $\mathcal{P}$ such that each voter ranks the candidates in increasing distance order.
- All ties are broken either lexicographically, in compliance with a fixed order $\succ_*$ over the candidates or randomly, by uniformly sampling the set of valid profiles.
- The most preferred candidate according to $\succ_*$ is denoted as c^*.
- Voter preferences are denoted by $\succ_i$ for every voter $i \in V$.
- Every candidate $c \in C$ has a fixed and predetermined preference order $\succ_c$ over the candidate set such that $c \succ_c c'$ for all $c' \neq c$.
- When random tie-breaking is used, we assume each candidate $c \in C$ has a fixed *utility function* $u_c : C \to \mathbb{R}$ over the possible winners of the election, subject to $u_c(a) > u_c(b) \Rightarrow a \succ_c b$.

Examples notation

- Voters are marked with large dots.
- Candidates are marked with lower case letters.
- Each candidate can position herself freely within the interval drawn beneath.



Voting Rules

- We discuss the following *irresolute* versions of voting rules, i.e. functions of the form $\mathcal{F} : \mathcal{L}(C)^n \to 2^C$ that map preference profiles to subsets of candidates.
- *Monotonic* positional scoring rules defined by $\alpha = (\alpha_m, \dots, \alpha_1)$ such that $\alpha_m \geq \dots \geq \alpha_1$, Plurality, in particular.
- Condorcet-consistent voting rules.
- *Super* Condorcet-consistent (SCC) voting rules — Condorcet-consistent voting rules that always produce the set of *Weak* Condorcet-winners, if it is nonempty.
- An RCG always has a Weak Condorcet-winner!

Best Responses

Lexicographic tie-breaking

- Let $\mathcal{F}$ be a voting rule that is computable in $O(T_{n,m})$ time for any preference profile of n voters over m candidates. For any candidate $c \in C$, the best responses set $\mathcal{B}_c(p, s)$ is computable in $O(n \cdot m \cdot [T_{n,m} + \log(m)])$ time, for any $p \in \mathbb{R}^n, s \in \mathbb{R}^m_\perp$.

Random tie-breaking

- Let $\mathcal{V}$ be the Plurality voting rule with random tie-breaking. For any voter and candidate position vectors $p \in \mathbb{R}^n$, $s \in \mathbb{R}^m_\perp$ and any given candidate $c \in C$, it is possible to compute

$$\max_{\succ_c} \{c' \in C \mid \exists s', Pr(\mathcal{V}(s'_c, s_{-c}) = c') > 0\}$$

in $O(poly(n, m))$ time.

Unrestricted Strategies

- Candidates may choose any position in $\mathbb{R}$.

Lexicographic tie-breaking

- For Condorcet-consistent voting rules when there is a single median position, SCC voting rules and monotonic scoring rules, a NE is only possible if c^* is the winner.
- For the same rules, for all $s \in \mathbb{R}^m_\perp$, there is $s'_{c^*} \in \mathbb{R}$ such that (s'_{c^*}, s_{-c^*}) is a NE.

Random tie-breaking

- For Condorcet-consistent voting rules when there is a single median position, SCC voting rules and monotonic scoring rules, for all $s \in \mathbb{R}^m_\perp$ and any candidate $c \in C$, there is $s'_c \in \mathbb{R}$ such that $Pr(\mathcal{V}(s'_c, s_{-c}) = c) > 0$.

Restricted Strategies with Lexicographic Tie-breaking

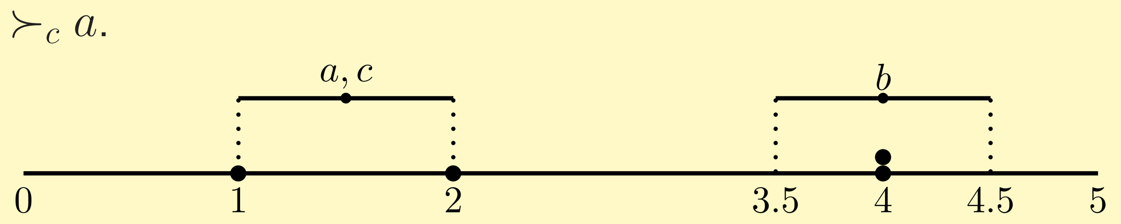
- Each candidate c may choose any position within a *closed* interval I_c.
- Ties are broken lexicographically.

Conditions of guaranteed equilibrium existence

Voting Rule	Withdrawals	Single Median Position	Number of Candidates
SCC	Yes	Any	Any
Condorcet-consistent	Any	Yes	Any
Monotonic scoring rule	Yes	Yes	Any
Plurality	Any	Any	2
Plurality	Yes	Any	3

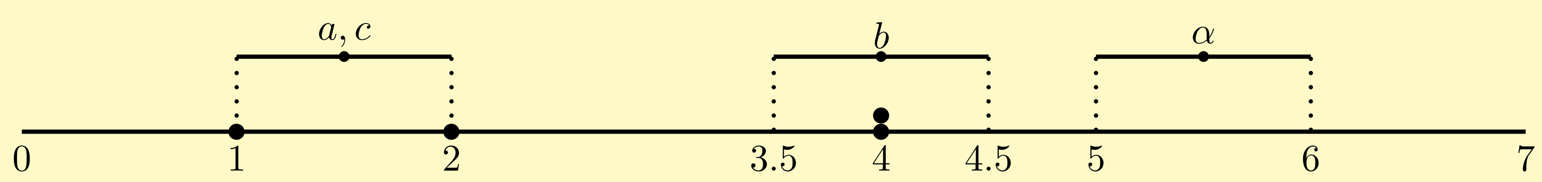
Example 1 without an equilibrium

- Plurality; no quitting; 3 or more candidates.
- Ties broken by $a \succ_* b \succ_* c$.
- Assume $b \succ_c a$.



Example 2 without an equilibrium

- Plurality; with or without quitting; 4 or more candidates.
- Ties broken by $\alpha \succ_* a \succ_* b \succ_* c$.
- Assume $b \succ_c a$ and $a, c \succ_b \alpha$.



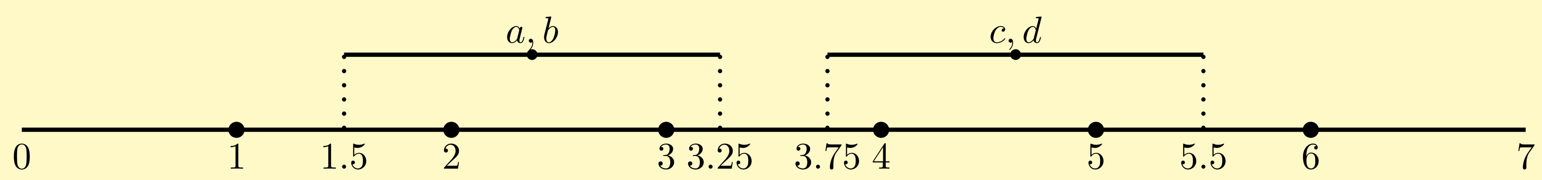
Restricted Strategies with Random Tie-breaking

- Each candidate c may choose any position within a *closed* interval I_c.
- Ties are broken randomly.
- Candidates wish to maximize *expected* utility.

Example 3 without an equilibrium

- Plurality; with or without quitting; 4 or more candidates.
- The utility functions are defined by

$$\forall x, y \in C, \quad u_x(y) = \begin{cases} 1 & if\ x = y \\ 0 & otherwise \end{cases}$$

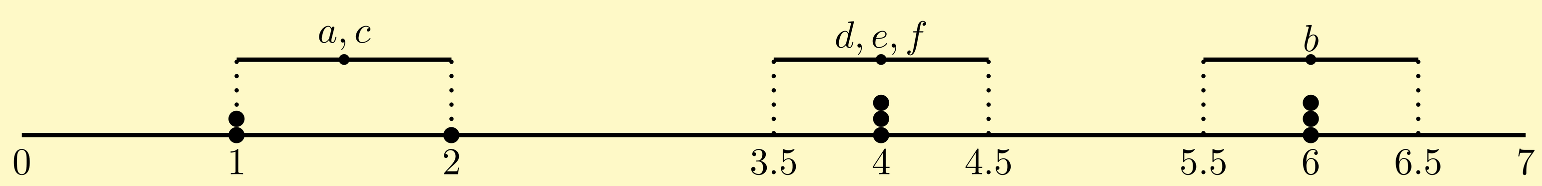


Example 4 without an equilibrium

- The voting rule is SCC with fallback (in case there are no Weak Condorcet-winners) to Plurality with lexicographic tie-breaking, subject to $a \succ_* b \succ_* c \succ_* d \succ_* e \succ_* f$.
- The utility functions are defined by

$$\forall x \in C, \quad u_c(x) = \begin{cases} 1 & if\ x = c\ or\ x = b \\ 0 & otherwise \end{cases}$$

$$\forall y \in C, y \neq c, \quad u_y(x) = \begin{cases} 1 & if\ x = y \\ 0 & otherwise \end{cases}$$



Hebrew University of Jerusalem, I-Core, Mobileye Vision Technologies Ltd.