Polynomial interpolation and counting lattice points in polytopes

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I have polytope (dimension d) and I want :

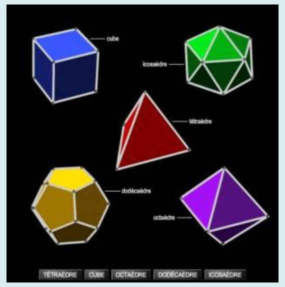
Compute volume of this polytope **OR** Compute lattice points in this polytope

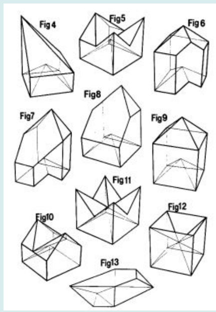
Compute polynomial of **degree d**

Algorithm?

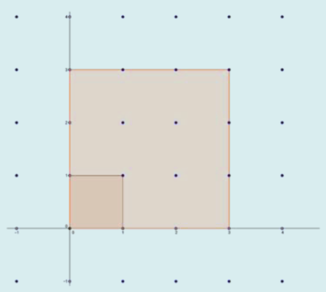
Integer dilates of polytopes

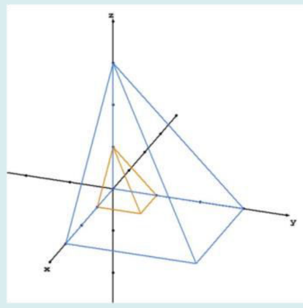
Convexe | Concave



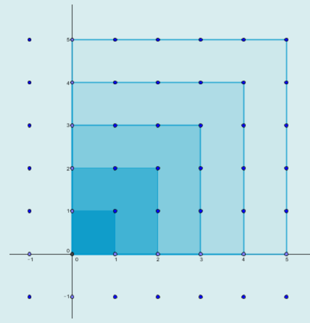


Square (2D) | Simplex (3D)





Lattice point in dilating unit square



- $L_p(1)$= Lattice point in the unit square C = 4
- $L_p(2)$= Lattice point in 2C = 9
- $L_p(3)$= Lattice point in 3C = 16

…

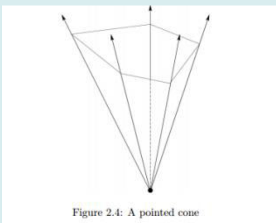
In general

- $L_p(n)=(n+1)^2=n^2+2n+1$

Polynomial interpolation : Need to compute a,b,c

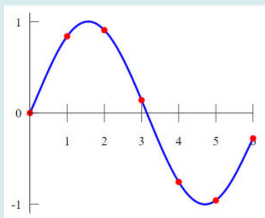
- $an^2+bn+c=4$
- $an2+bn+c=9$
- $an2+bn+c=16$

Barvinok



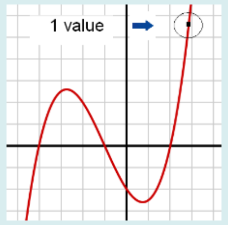
The most common method Use power series

Clauss



Clauss method is based on the polynomial interpolation : For polytope of dimansion d we need to compute **d+1 values**

Our approach

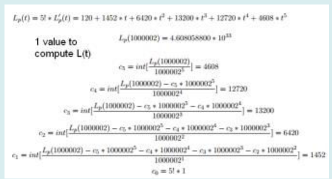


Our method is based on the polynomial interpolation : For polytope of dimansion d **1 value** or just an **approximate** of this value is sufficient to generate (Ehrhart) polynomial

Ehrhart theorem

if P is a polytope, and nP is the polytope formed by expanding P by a factor of n in each dimension, then L(n) is the number of integer lattice points in nP. Ehrhart showed in 1962 that L is a rational polynomial of degree d in n :

$$L_p(n)=a_d n^d + a_{d-1}n^{d-1} + a_{d-2}n^{d-2} + ... + a_1 n^1 + a_0 \;\; ; \;\; a_i = c_i / d! \;\; avec \;\; c_i \in Z$$



First point

Lower and upper bound on the coefficients of Ehrhart polynomial

- Upper bound Ulrich Betke et Peter McMullen (1985)
- Lower bound Martin Henk et Makoto Tagami (2009)

$$c_r < d!\left((-1)^{d-r} stirl(d,r) vol(P) + (-1)^{d-r-1}\frac{stirl(d,r+1)}{(d-1)!}\right) = \alpha_r$$

$$c_r > d!\left(\frac{1}{d!}\left((-1)^{d-r} stirl(d+1,r+1) + (d! vol(P) - 1) M_{r,d}\right)\right) = \beta_r$$

Second point

$$\frac{L_p(n)}{n^d} = c_d + \underbrace{\frac{c_{d-1}n^{d-1} + c_{d-2}n^{d-2} + ... + c_1 n^1 + c_0}{n^d}}_{\lim_{n\to\infty} \to 0}$$

$$\left[\frac{L_p(k)}{k^d}\right] = c_d + \left[\frac{c_{d-1}k^{d-1} + c_{d-2}k^{d-2} + ... + c_1 k^1 + c_0}{k^d}\right] = c_d = vol(P)$$

General formula

Lemma 1.1. *Let $\alpha_r \in Z$, we define $f_{\alpha,d-1}$ as :*

$$f_{\alpha,d-1}(t) = \frac{\sum_{r=0}^{d-1}\alpha_r t^r}{t^d}$$

So :

- $\lim_{t\to\infty} f_{\alpha,d-1}(t) = 0$
- $\exists k_{\alpha,d-1} \in N$ as $\forall t > k_{\alpha,d}$ then $|f_{\alpha,d-1}(t)| < \frac{1}{2}$

Lemma 1.2. *Let $(\beta_r, c_r, \alpha_r) \in Z^3$ as $\beta_r \le c_r \le \alpha_r$. Let $f_{\alpha,d}$, $f_{\beta,d}$ et $f_{c,d}$ function define in lemma 1.1. we define $k_{max,d-1}$, k_{max} as :*

- $k_{max,d-1} = Max\{k_{\alpha,d-1}, k_{\beta,d-1}\}$
- $k_{max} = Max\{k_{max,r} \, \forall r \in [0, d-1]\}$

So $\forall t > k_{max}$, $\forall r \in [0, d-1]$ we have :

$$\lfloor f_{c,r}(t) \rceil = 0$$

Maximum margin of error of our aproximation. We can control this uper bound. If we add another approximation this bound increase.

Theorem 1.3. *Let $(\beta_r, c_r, \alpha_r) \in Z^3$ as $\beta_r \le c_r \le \alpha_r$. Let $P(t)$ an integer polynomial :*

$$P(t) = \sum_{r=0}^{d} c_r t^r$$

Combining lemma 1.1 and 1.2 we have :

- $c_d = \lfloor \frac{P(k_{max})}{k_{max}^d} \rceil$
- $c_r = \lfloor \frac{P(k_{max}) - \sum_{i=r+1}^{d} c_i k_{max}^i}{k_{max}^r} \rceil \; \forall r \in [0, d-1]$

Theorem 1.4 (Approximation). *Let $X \in R$, if :*

$$|X - P(k_{max})| \le \frac{\sum_{r=0}^{d-1}(\alpha_r + \beta_r)k_{max}^r}{2}$$

So :

- $c_d = \lfloor \frac{X}{k_{max}^d} \rceil$
- $c_r = \lfloor \frac{X - \sum_{i=r+1}^{d} c_i k_{max}^i}{k_{max}^r} \rceil \; \forall r \in [0, d-1]$

Starting from only one estimate $P(k_{max})$, we can generate the Ehrhart polynomial.

Starting from an approximation of $P(k_{max})$, we can generate the Ehrhart polynomial.

Parametric polytope : We need just one value (or approximation) to generate any parametric multivariate polynomials.

Theorem 6.1. *Let $(\beta_r, c_r, \alpha_r) \in Z^3$ as $\beta_r \le c_r \le \alpha_r$. Let $P(n,m)$ an integer parametric polynomial :*

$$P(n,m) = \sum_{r=0}^{p}\sum_{j=0}^{q} c_{i,j} n^r m^j = \sum_{r=0}^{p} g_i(m) n^i$$

$$P(n,m) = (\sum_{j=0}^{q} c_{p,j} m^j) n^p + (\sum_{j=0}^{q} c_{p-1,j} m^j) n^{p-1} + ... + (\sum_{j=0}^{q} c_{0,j} m^j)$$

We can have equivalent to lemma 1.1 and 1.2 for parametric polynomial so :

- $g_p(m) = \lfloor \frac{P(k_{max,m})}{k_{max}^p} \rceil$
- $g_r(m) = \lfloor \frac{P(k_{max,m}) - \sum_{i=r+1}^{p} g_r(m) k_{max}^i}{k_{max}^r} \rceil \; \forall r \in [0, p-1]$
- $c_{r,q} = \lfloor \frac{g_q(k'_{max})}{k'^d_{max}} \rceil$
- $c_{r,j} = \lfloor \frac{g_j(k'_{max}) - \sum_{i=j+1}^{q} c_{j,q} k'^i_{max}}{k'^r_{max}} \rceil \; \forall j \in [0, q-1]$

Work in progress.
(This example : 2 variable. General formula for p variables looks like this.)

Illustration source : Wikipedia, Google, Hele ne Rochais