Axiomatic Foundations of Voting Theory (part I)

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Computational Social Choice Summer School
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18-22 July 2016 COST IC1205

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- Election with 3 candidates
  a, b, c for mayor of a town
- 303 voters
- Each voter casts a ballot

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102	101	100
a	***b***	***c***
b	***c***	***b***
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- For 3 or more candidates, plurality ≠ majority

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- Elect US senator from NY State
- 3-way 1980 vote

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- Such examples are major reason for opposition to plurality rule . . .
- . . . and interest in voting theory

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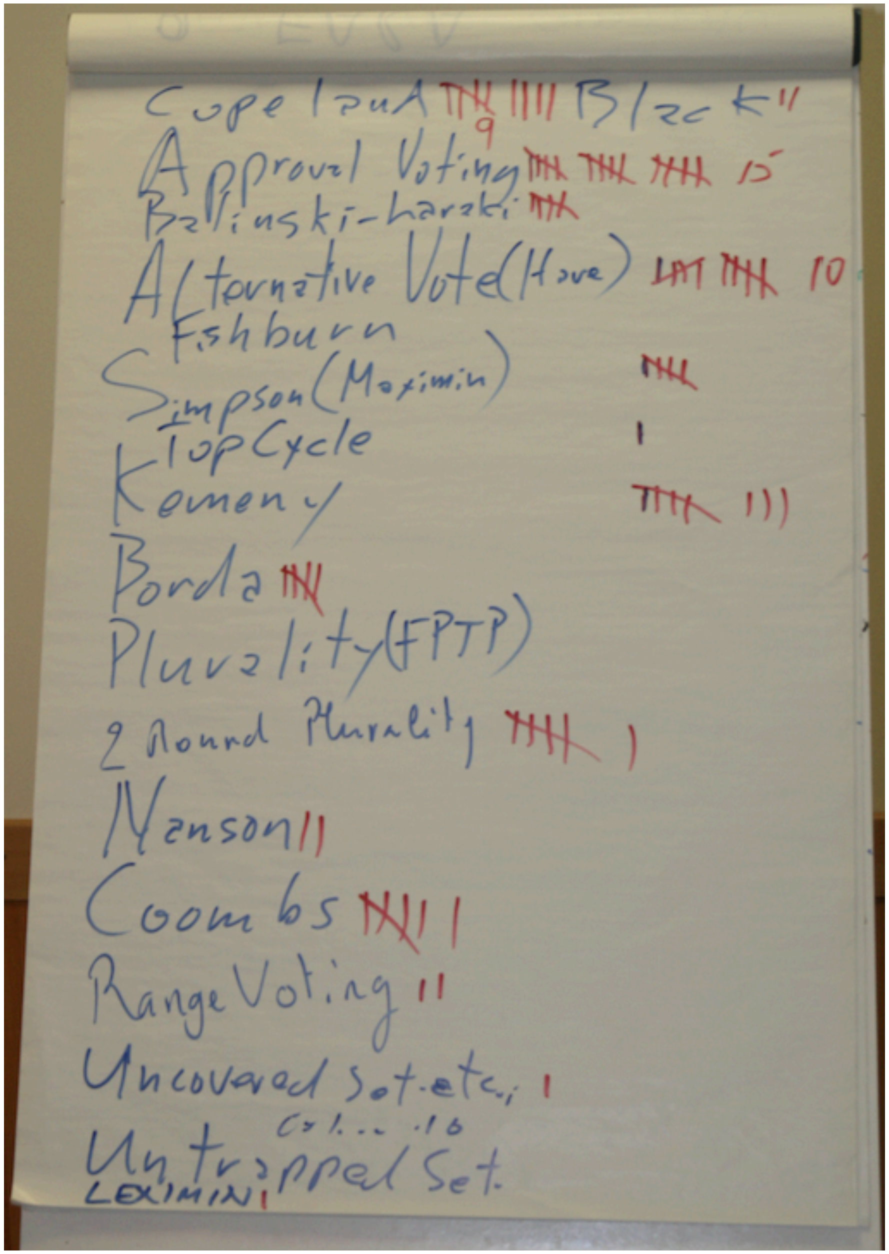
- 2010 Meeting, Ch. du Baffy, Normandy
- We agreed . . . on almost nothing . . . *



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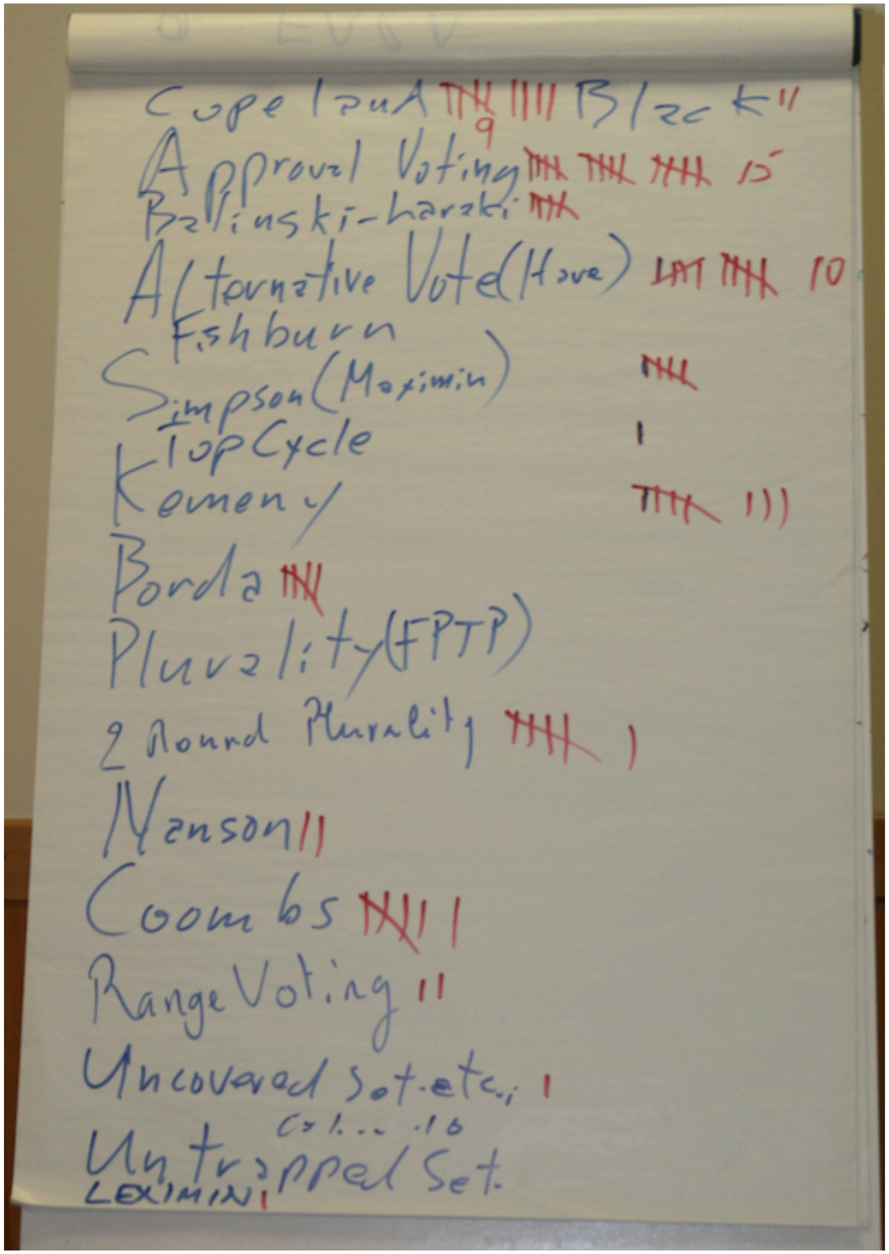

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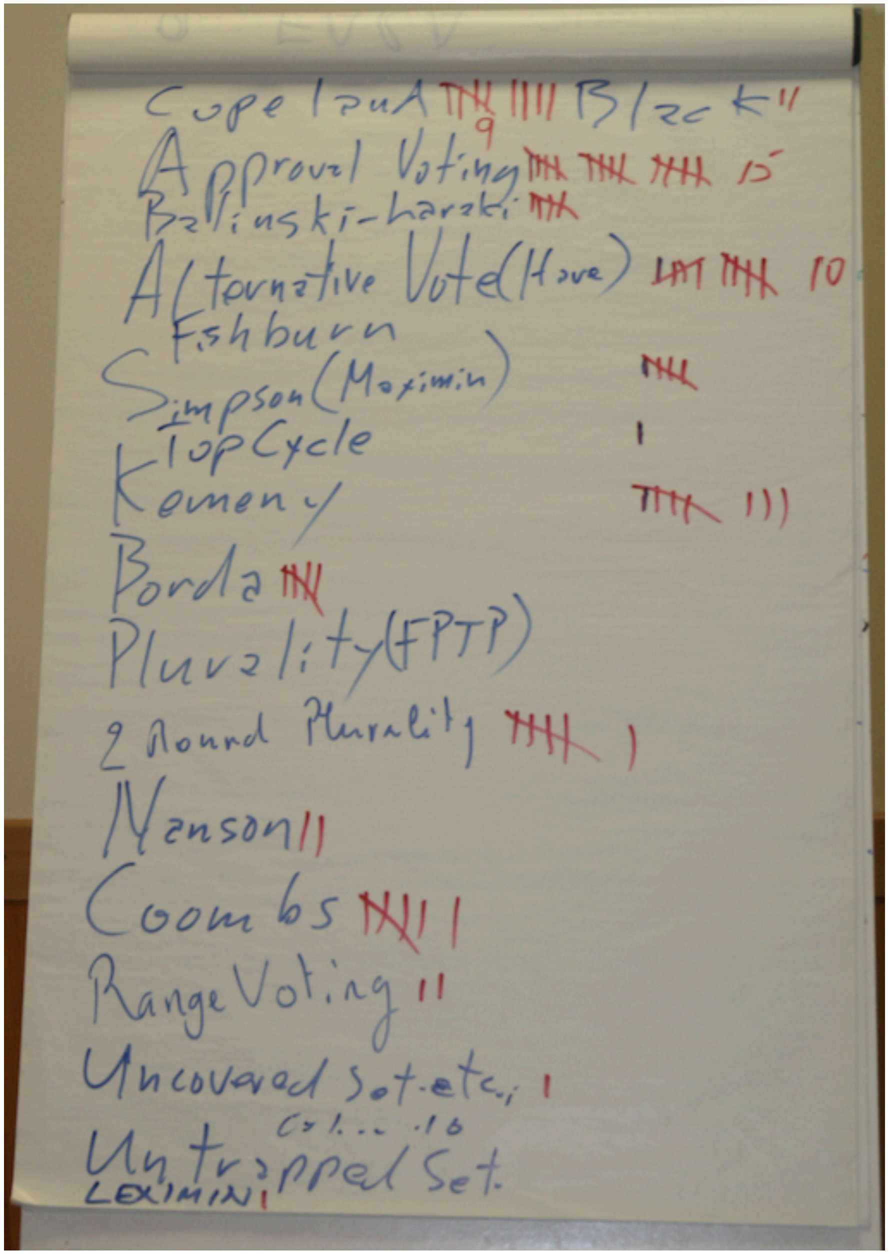

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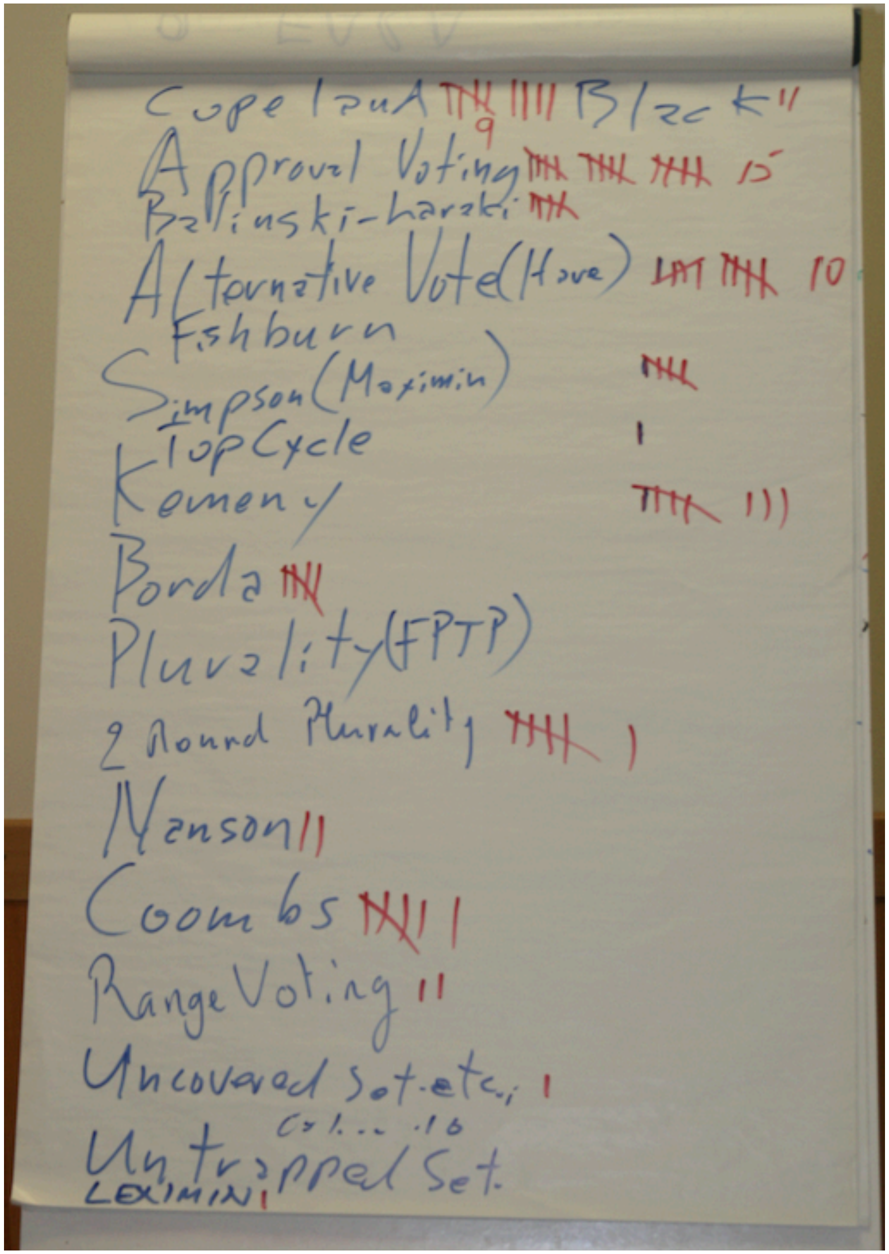

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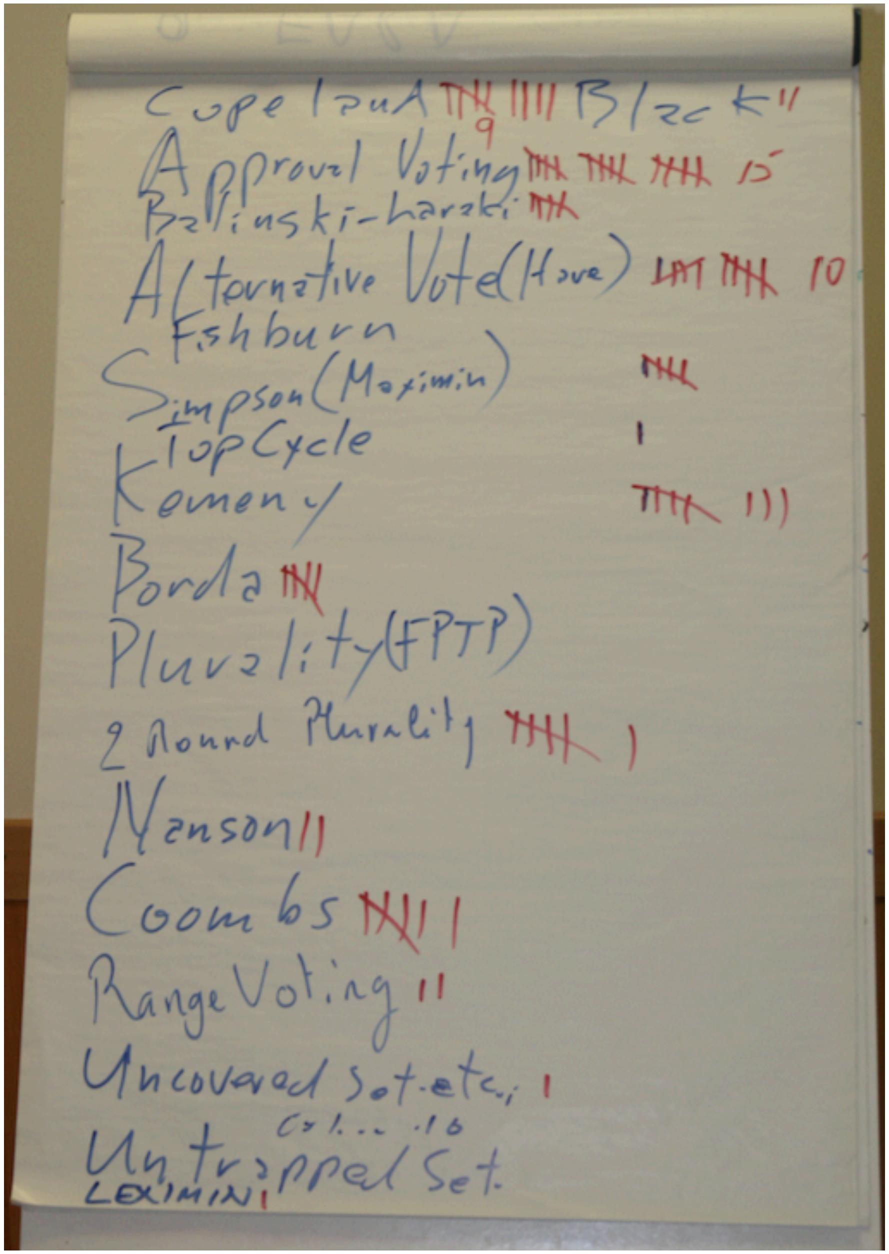

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- . . . *but note score for plurality is 0
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- Which voting rule won?
- What question should you be asking me . . .?

1) Intro: Three voting rules

P_2

102	101	100	1
a	***b***	***c***	***c***
b	***c***	***a***	***b***
c	***a***	***b***	***a***

1) Intro: Three voting rules

- In profile P_2
  - 202 voters rank a over b

P_2

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1) Intro: Three voting rules

- In profile P_2
  - ➢ 202 voters rank a over b
  - ➢ 102 rank b over a
  - ➢ $Net_{P2}(a > b) = 202 - 102 = 100$

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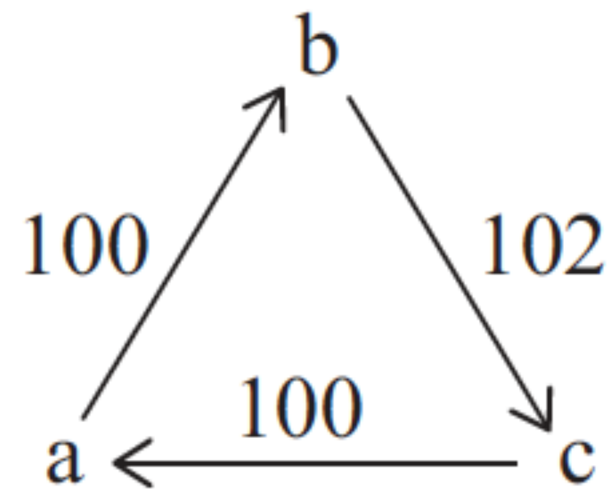

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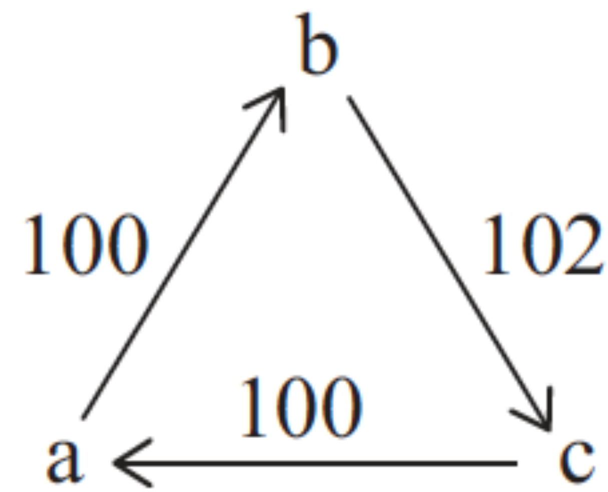

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  - A graph in which the vertices are the candidates
  - For each two vertices, *either* a → b *or* a ← b is an edge

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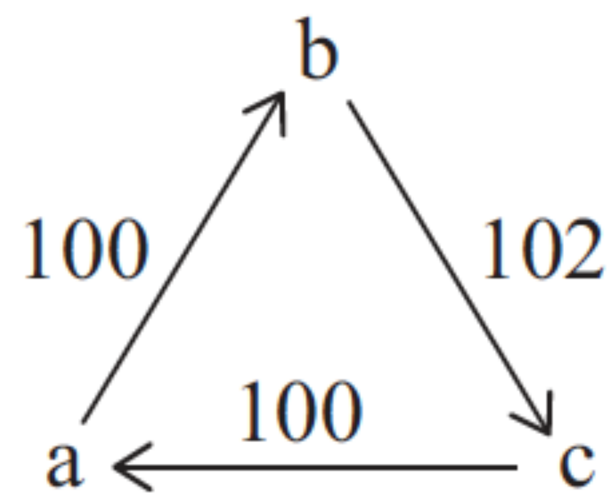

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- We get a ***weighted tournament*** induced by the profile P_2
- ***Tournament***:
  - ➢ A graph in which the vertices are the candidates
  - ➢ For each two vertices, *either* $a \rightarrow b$ *or* $a \leftarrow b$ is an edge
- ***Edge weights***:
  - ➢ Assign $Net_{P2}(a > b)$ to $a \rightarrow b$

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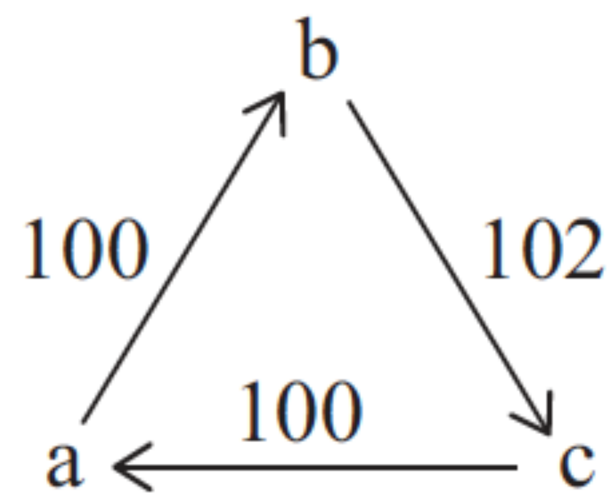

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Pairwise Majority Preference

- x $>^{\mu}$ y means (strictly) more voters rank x over y than y over x

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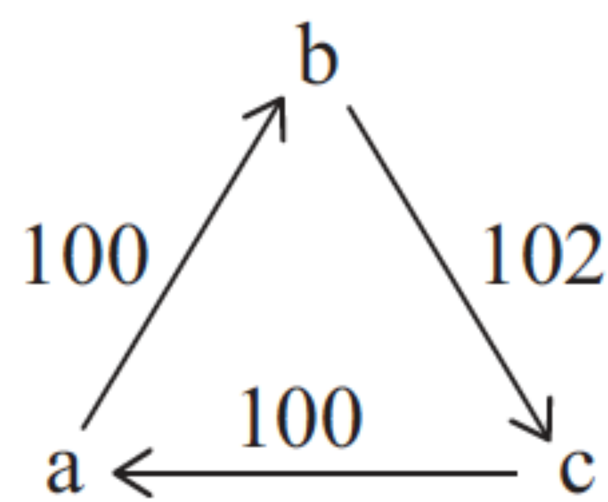

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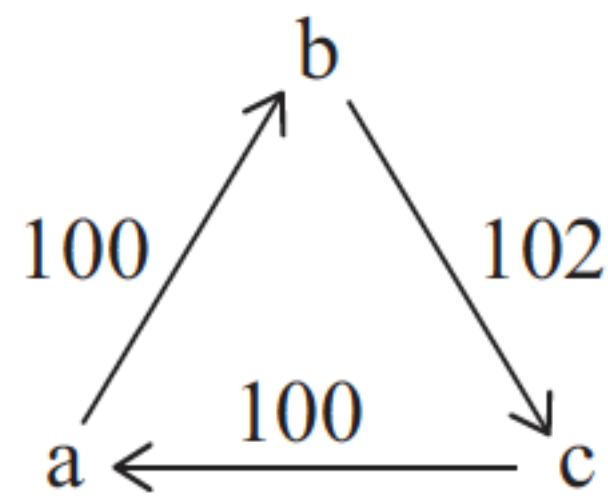

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  - ➢ $>^{\mu}$ is ***intransitive***

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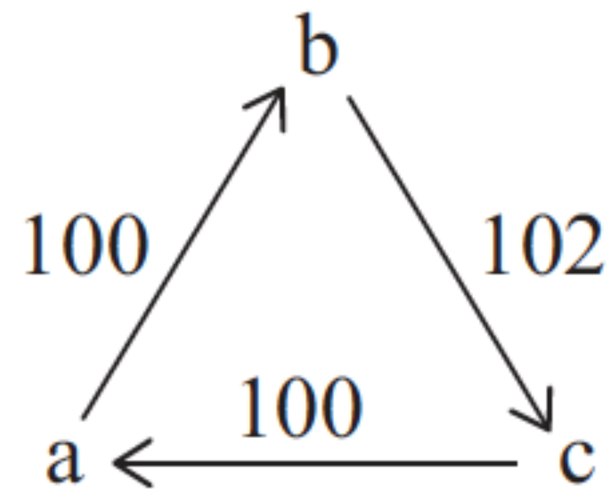

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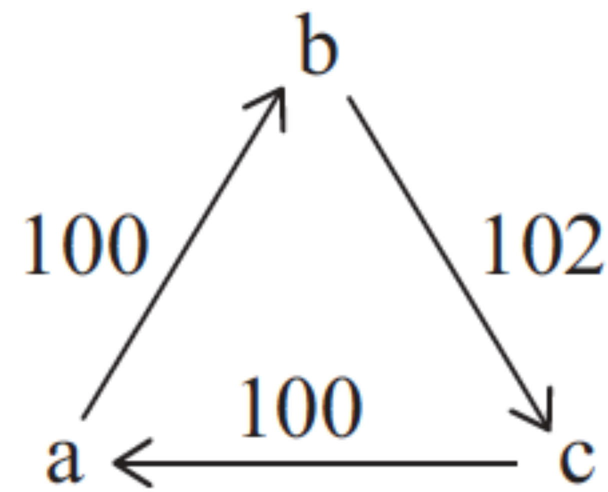

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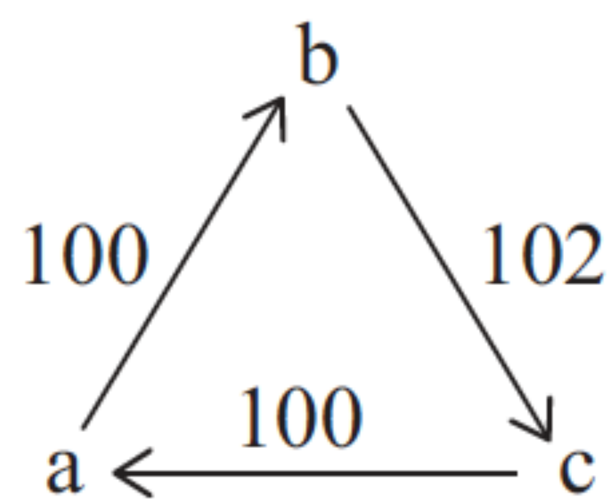

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- Symmetric Copeland Score

$$Cop(x) = |\{y \mid x >^{\mu} y\}| - |\{y \mid y >^{\mu} x\}|$$

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  $0 = Cop(b) = Cop(c)$ **Tie!**

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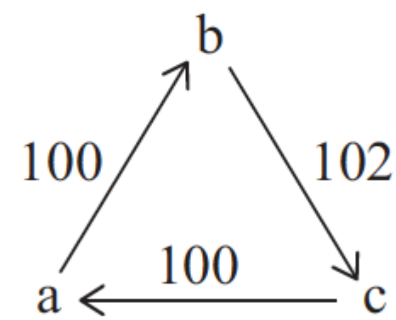

1) Intro: Three voting rules

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- $x >^{\mu} y$ means (strictly) more voters rank x over y than y over x
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- For P_2, we have $a >^{\mu} b >^{\mu} c >^{\mu} a$
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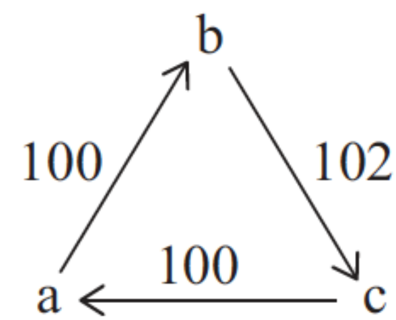

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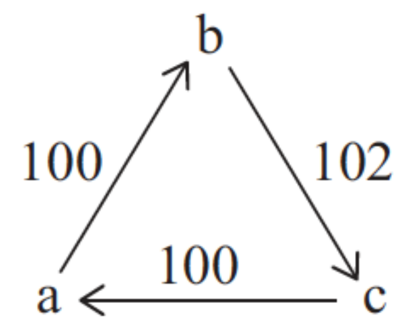

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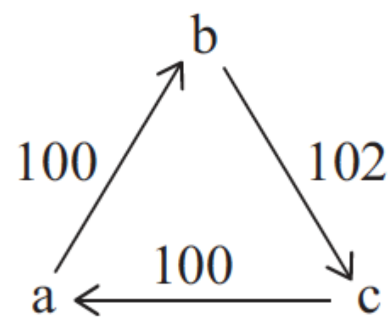
Exercise 1: Do other versions of Copeland score yield the same rule?



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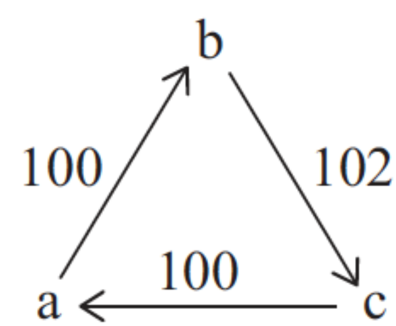

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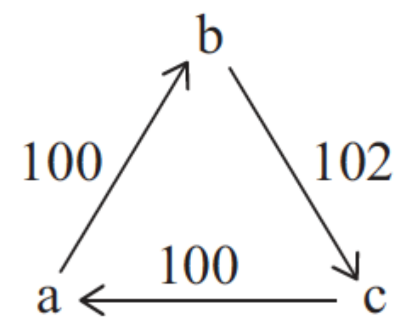

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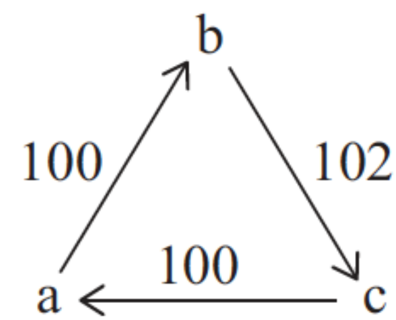

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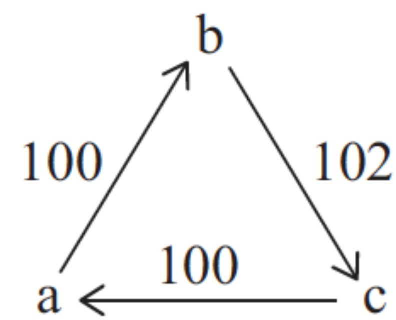

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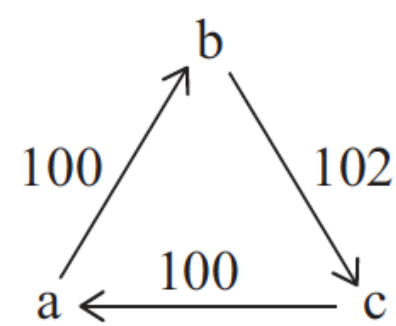

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- Standard Borda weights:
  m-1, m-2, ..., 0; **evenly**
  2, 1, 0 for m = 3 **spaced**
- "Symmetric" Borda wts
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  yield exactly $\beta(x)$

1) Intro: Three voting rules

Exercise 2

a) Scoring vectors $w_1, \ldots, w_m$ and $v_1, \ldots, v_m$ are ***affinely equivalent*** if there exist constants γ, δ with γ > 0 and $v_i = \gamma w_i + \delta$ for each i. Show that

- affinely equivalent vectors induce same voting rule, and
- any two evenly spaced vectors are affinely equivalent.

b) Show symmetric Borda weights yield a total score = β(x).

Scoring Rules

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- "Symmetric" Borda wts m-1, m-3, ... , -(m-1) yield exactly β(x)

2) Social Choice Functions

Goal: select one alternative
from a finite set A

1. Each voter (finitely many) casts a ballot
2. Apply some voting rule

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But ties are possible!

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Alternatives = . . . ?

- **candidates for mayor of small town**
- € budgets for new firehouse
- Estimates for amount of oil lying beneath a region
- (amend the constitution?) yes or no
- different versions of an immigration reform bill
- committees

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A ballot might be . . .

- an individual alternative
- a strict ranking of alternatives

<u>Francine</u>

d

a

c

b

e

linear ordering $\geq_F$ of A = {a,b,c,d,e}

$\mathcal{L}$(A) = the set of all possible linear orderings of A. $|\mathcal{L}(A)| = m!$

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Ahmed
d,e
c
a,b

d $\geq_A$ e and e $\geq_A$ d ***both*** hold, so we say "Ahmed is indifferent to d and e." **But maybe not . . .**

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- a separate score (1-10) assigned to each alternative

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There are many types of voting.

We focus on one type:
Social Choice Functions ***SCFs***

2) Social Choice Functions

- $N = \{1, 2, . . ., n\}$ set of n voters
- A = finite set of m alternatives
- $\mathcal{C}(A) = \{X \subseteq A \mid X \neq \varnothing\}$
- $\geq_j$ = ballot cast by voter j, an element of $\mathcal{L}(A)$
- $P = (\geq_1, \geq_2, . . ., \geq_n) \in \mathcal{L}(A)^n$ specifies a ballot for each voter $j \in N$. P is a ***profile.***
- A ***SCF*** is a function that assigns, to each election, one winner (or several, if a tie) $f : \mathcal{L}(A)^n \rightarrow \mathcal{C}(A)$
- A SCF with no ties is ***resolute***
- A ***variable electorate*** SCF handles profiles for all finite n

$$\mathcal{L}(A)^{<\infty} = \bigcup \{\mathcal{L}(A)^n \mid n \in \mathbf{N}\}$$

$$f : \mathcal{L}(A)^{<\infty} \rightarrow \mathcal{C}(A)$$

3) A taste of strategic manipulation

Consider profile P_3, in which Ali is one of the last 2 voters.

P_3

2	3	2
e	*d*	*a*
c	*e*	*b*
a	*b*	*c*
d	*c*	*d*
b	*a*	*e*

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e loses to d and to no one else

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d beats b, e by 5-2

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- By misrepresenting her preferences Ali does better

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- How much better**?**
- We don't know – cannot extract cardinal utilities from ordinal preferences.

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a	*b*	*c*
d	*c*	*d*
b	*a*	*e*

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- By misrepresenting her preferences Ali does better – she has ***manipulated*** the election
- How much better**?**
- We don't know – cannot extract cardinal utilities from ordinal preferences.
- In P_3*, d beats each other alternative in the pairwise majority sense; d is a ***Condorcet alternative***

P_3

2	3	2
e	d	a
c	e	b
a	b	c
d	c	d
b	a	e

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- A SCF honoring this principle is called a ***Condorcet extension***

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P_3

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a	*b*	*c*
d	*c*	*d*
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P_3

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EXERCISE 3 Show Borda can never be manipulated via reversal

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e	*d*	*a*
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- In P_3, can Ali manip'te Borda?
- Yes: lift d to top, push others down. e: (6) d: (4)→(10)

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- v's ballot $\geq_v$ in P = his sincere ranking

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- $f(P^*) >_v f(P)$? the attempt succeeds: according to his sincere ranking $\geq_v$, he strictly prefers outcome from insincere ballot

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If a rule <u>always</u> yields ties, it is never s-v manipulable

4) The GST: Gibbard-Satterthwaite Theorem and Arrow's Impossibility Theorem

4a) The GST: Gibbard-Satterthwaite Theorem

Theorem (Alan Gibbard, Mark Satterthwaite)
Let f be any SCF for three or more alternatives.

If f is:

- resolute
- nonimposed
- and strategyproof

then f must be a dictatorship

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then f must be a dictatorship – winner is dictator's top-ranked alternative

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Theorem (Kenneth Arrow) Let f be any Social Welfare Function (SWF) for **three or more alternatives**.

Social Welfare Function

- Ballots are linear orders of A (as before), . . .
- but the outcome F(P) of an election is a weak order of A

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- and **independence of irrelevant alternatives "IIA"**

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A SWF F satisfies the Weak Pareto property if

- whenever each voter i ranks $x >_i y$,
- the outcome under F has $x > y$

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F respects unanimity in strict preferences

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A SWF F satisfies **IIA** if

- for each pair x, y of alternatives
- the relative ranking of x VS y in the outcome F(P)
- depends only on the relative ranking of x VS y in the ballots

4b) Arrow's Impossibility Theorem

Example P	Robert	Sandra	Dieter	Pablo	Mei-Ling
	x	x	x	y	y
	y	y	y	x	x
	a	a	a	a	a
	b	b	b	b	b
	c	c	c	c	c

4b) Arrow's Impossibility Theorem

Example P

Robert	Sandra	Dieter	Pablo	Mei-Ling
x	x	x	y	y
y	y	y	x	x
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c

P*

Robert	Sandra	Dieter	Pablo	Mei-Ling
x	x	x	y	y
y	y	y	a	a
a	a	a	b	b
b	b	b	c	c
c	c	c	x	x

4b) Arrow's Impossibility Theorem

Example P

Robert	Sandra	Dieter	Pablo	Mei-Ling
x	x	x	y	y
y	y	y	x	x
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c

P*

Robert	Sandra	Dieter	Pablo	Mei-Ling
x	x	x	y	y
y	y	y	a	a
a	a	a	b	b
b	b	b	c	c
c	c	c	x	x

$P \mapsto P^*$: no individual voter changes **relative** ranking of x VS y.

4b) Arrow's Impossibility Theorem

Example P	Robert	Sandra	Dieter	Pablo	Mei-Ling
	x	**x**	**x**	**y**	**y**
	y	**y**	**y**	**x**	**x**
	a	a	a	a	a
	b	b	b	b	b
	c	c	c	c	c

P^*	Robert	Sandra	Dieter	Pablo	Mei-Ling
	x	**x**	**x**	**y**	**y**
	y	**y**	**y**	a	a
	a	a	a	b	b
	b	b	b	c	c
	c	c	c	**x**	**x**

$P \mapsto P^*$: no individual voter changes **relative** ranking of **x** VS **y**.
So IIA says "If F(P) = **x** > **y** > a > b > c then F(P*) must have **x** > **y**"

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Equivalently . . . **No** SWF for 3 or more alternatives satisfies weak Pareto + **IIA** + nondictatoriality

5) Axioms I Pareto property, anonymity, neutrality

- A profile of 201 voters

101	100
b	**a**

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- A profile of 201 voters
- For the moment, only top choices visible

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5) Axioms I Pareto property, anonymity, neutrality

- A profile of 201 voters
- For the moment, only top choices visible
- Based on this limited info, **b** wins

101	100
b	**a**

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- For the moment, only top choices visible
- Based on this limited info, **b** wins
- Argument for b is stronger than "b is the plurality winner" . . .

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- A profile of 201 voters
- For the moment, only top choices visible
- Based on this limited info, **b** wins
- Argument for b is stronger than "b is the plurality winner" . . . ***WHY?***

101	100
b	**a**

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- For the moment, only top choices visible
- Based on this limited info, **b** wins
- Argument for b is stronger than "b is the plurality winner" . . . ***WHY?*** Majoritarian Principle

101	100
b	**a**

5) Axioms I Pareto property, anonymity, neutrality

- A profile of 201 voters
- For the moment, only top choices visible
- Based on this limited info, **b** wins
- We see the hidden info

101	100
b	**a**
a	d
c	e
d	f
e	**b**
f	c

5) Axioms I Pareto property, anonymity, neutrality

- A profile of 201 voters
- For the moment, only top choices visible
- Based on this limited info, **b** wins
- We see the hidden info
- Should **b** still win?

101	100
b	**a**
a	d
c	e
d	f
e	**b**
f	c

5) Axioms I Pareto property, anonymity, neutrality

- A profile of 201 voters
- For the moment, only top choices visible
- Based on this limited info, **b** wins
- We see the hidden info
- Should **b** still win?
- Or should it be **a**?

101	100
b	**a**
a	d
c	e
d	f
e	**b**
f	c

5) Axioms I Pareto property, anonymity, neutrality

- A profile of 201 voters
- For the moment, only top choices visible
- Based on this limited info, **b** wins
- We see the hidden info
- Should **b** still win? **hands**
- Or should it be **a**?

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b	**a**
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d	f
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- Should **b** still win? **hands**
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b is Condorcet alt, so b wins Copeland

<u>101</u>	<u>100</u>
b	**a**
a	d
c	e
d	f
e	**b**
f	c

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b is Condorcet alt, so b wins Copeland

a is Borda winner. . . Borda "favors compromise over Maj."

101	100
b	**a**
a	d
c	e
d	f
e	**b**
f	c

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- For the moment, only top choices visible
- Based on this limited info, **b** wins
- We see the hidden info
- Should **b** still win? **hands**
- Or should it be **a**? **hands**

b is Condorcet alt, so b wins Copeland

a is Borda winner. . . Borda "favors compromise over Maj."

<u>101</u>	<u>100</u>
b	**a**
a	d
c	e
d	f
e	**b**
f	c

5) Axioms I Pareto property, anonymity, neutrality

- A profile of 201 voters
- Someone says it is c, not a or b, who should win

101	100
b	a
a	d
c	e
d	f
e	b
f	**c**

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- Someone says it is c, not a or b, who should win
- Counterargument is ... ?

101	100
b	a
a	d
c	e
d	f
e	b
f	**c**

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101	100
b	a
a	d
c	e
d	f
e	**b**
f	**c**

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- Someone says it is c, not a or b, who should win
- Counterargument is ... ?
- Every voter prefers b to c

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b	a
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e	**b**
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- c is "Pareto dominated" by b

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e	**b**
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- A profile of 201 voters
- Someone says it is c, not a or b, who should win
- Counterargument is ... ?
- Every voter prefers b to c
- c is "Pareto dominated" by b

Axiom An SCF f satisfies the ***Pareto Principle*** if f(P) never includes a Pareto dominated alternative

101	100
b	a
a	d
c	e
d	f
e	**b**
f	**c**

5) Axioms I Pareto property, anonymity, neutrality

Easy Theorem: *Pareto Principle* is satisfied by

- Plurality rule
- Borda
- Copeland

101	100
b	a
a	d
c	e
d	f
e	**b**
f	**c**

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Easy Theorem: *Pareto Principle* is satisfied by

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(And by most "reasonable" SCFs)

101	100
b	a
a	d
c	e
d	f
e	**b**
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Easy Theorem: *Pareto Principle* is satisfied by

- Plurality rule
- Borda
- Copeland

(And by most "reasonable" SCFs)

Pareto implies that winner for this 201-vote profile is **a** or **b**

101	100
b	a
a	d
c	e
d	f
e	**b**
f	**c**

5) Axioms I Pareto property, anonymity, neutrality

Assume the winner for this profile is **a**.

101	100
b	a
a	d
c	e
d	f
e	b
f	c

5) Axioms I Pareto property, anonymity, neutrality

Assume the winner for this profile is **a**.

- Anna is a type **I** voter (among 1st group of 101); Stevo is type **II**

101	100
b	a
a	d
c	e
d	f
e	b
f	c

5) Axioms I Pareto property, anonymity, neutrality

Assume the winner for this profile is **a**.

- Anna is a type **I** voter (among 1st group of 101); Stevo is type **II**
- Anna is convinced by Sara to change her ballot to type **II**.

101	100
b	a
a	d
c	e
d	f
e	b
f	c

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Assume the winner for this profile is **a**.

- Anna is a type **I** voter (among 1st group of 101); Stevo is type **II**
- Anna is convinced by Sara to change her ballot to type **II**.
- At same time Stevo is convinced to change his ballot to type **I**

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b	a
a	d
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101	100
b	a
a	d
c	e
d	f
e	b
f	c

After both switches, how should outcome change?

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After both switches, how should outcome change?

It depends!

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b	a
a	d
c	e
d	f
e	b
f	c

After both switches, how should outcome change?

It depends! In some contexts, ***not at all***.

5) Axioms I Pareto property, anonymity, neutrality

Axiom An SCF f is ***anonymous*** if each pair of voters play interchangeable roles: f(P) = f(P*) whenever P* is obtained from P by swapping ballots of 2 voters.

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Math This says $f(P) = f(\tau P)$ for each ***transposition*** τ of voters.

Transpositions generate the full symmetric group. So $f(P) = f(\sigma P)$ for each ***permutation*** σ of the set N of voters.

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Axiom An SCF f is ***anonymous*** if each pair of voters play interchangeable roles: f(P) = f(P*) whenever P* is obtained from P by swapping ballots of 2 voters.

Anonymity is a very ***strong*** form of equal influence by voters. Non-dictatoriality is a very ***weak*** form.

Math This says $f(P) = f(\tau P)$ for each ***transposition*** τ of voters.

Transpositions generate the full symmetric group. So $f(P) = f(\sigma P)$ for each ***permutation*** σ of the set N of voters.

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101	100
b	a
a	d
c	e
d	f
e	b
f	c

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Again, assume the winner for this profile is **a**.

- This time, switch *candidate* a with *candidate* f (in all ballots)

101	100
b	a
a	d
c	e
d	f
e	b
f	c

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- This time, switch *candidate* a with *candidate* f (in all ballots)

101	100
b	a **f**
a **f**	d
c	e
d	f **a**
e	b
f **a**	c

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101	100
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f	d
c	e
d	**a**
e	b
a	c

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Again, assume the winner for this profile is **a**.

- This time, switch *candidate* a with *candidate* f (in all ballots)
- After the switches, how should outcome change?
- Assume the voting rule treats candidates equivalently.

101	100
b	**f**
f	d
c	e
d	**a**
e	b
a	c

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Again, assume the winner for this profile is **a**.

- This time, switch *candidate* a with *candidate* f (in all ballots)
- After the switches, how should outcome change?
- Assume the voting rule treats candidates equivalently.
- **f** should win, post switch

101	100
b	**f**
f	d
c	e
d	**a**
e	b
a	c

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- This time, switch *candidate* a with *candidate* f (in all ballots)
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Axiom An SCF f is ***neutral*** if each pair of candidates play interchangeable roles.

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Axiom An SCF f is ***neutral*** if each pair of candidates play interchangeable roles: $f(P^{\tau}) = \tau[f(P)]$ whenever P^{τ} is obtained from P by swapping 2 alternatives in all ballots.

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Axiom An SCF f is ***neutral*** if each pair of candidates play interchangeable roles: $f(P^{\tau}) = \tau[f(P)]$ whenever P^{τ} is obtained from P by swapping 2 alternatives in all ballots.

Again, we can replace τ with σ: $f(P^{\sigma}) = \sigma^{-1}[f(P)]$

Why use ***inverse*** of σ**?**

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- These three axioms are easy to satisfy: many rules satisfy all of them

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- But they already show ***you can't always get what you want***

And certainly, the GST and Arrow's Theorem show this

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- But they already show ***you can't always get what you want***
- Together, they have negative implications for resoluteness

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- But they already show ***you can't always get what you want***
- Together, they have negative implications for resoluteness
- A profile for 3k voters, m alternatives

k	k	k
a	c	b
b	a	c
c	b	a

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- A profile for 3k voters, m alternatives

k	k	k
a	c	b
b	a	c
c	b	a
x_1	x_1	x_1
$\vdots$	$\vdots$	$\vdots$
x_{m-2}	x_{m-2}	x_{m-2}

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a	c	b
b	a	c
c	b	a
x_1	x_1	x_1
⋮	⋮	⋮
x_{m-2}	x_{m-2}	x_{m-2}

We'll show a 3-way tie is forced

5) Axioms I Pareto property, anonymity, neutrality

- Pareto $\Rightarrow f(P) \subseteq \{a,b,c\}$

k	k	k
a	c	b
b	a	c
c	b	a
x_1	x_1	x_1
$\vdots$	$\vdots$	$\vdots$
x_{m-2}	x_{m-2}	x_{m-2}

We'll show a 3-way tie is forced

5) Axioms I Pareto property, anonymity, neutrality

- Pareto $\Rightarrow$ $f(P) \subseteq \{a,b,c\}$
- WLOG assume $a \in f(P)$

k	k	k
a	c	b
b	a	c
c	b	a
x_1	x_1	x_1
⋮	⋮	⋮
x_{m-2}	x_{m-2}	x_{m-2}

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- Pareto $\Rightarrow f(P) \subseteq \{a,b,c\}$
- WLOG assume $a \in f(P)$
- First, permute voters

k	k	k
a	c	b
b	a	c
c	b	a
x_1	x_1	x_1
⋮	⋮	⋮
x_{m-2}	x_{m-2}	x_{m-2}

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- First, permute voters
- ρ: 1st k $\rightarrow$ last k $\rightarrow$ **mid k**

k	k	k
a	c	b
b	a	c
c	b	a
x_1	x_1	x_1
$\vdots$	$\vdots$	$\vdots$
x_{m-2}	x_{m-2}	x_{m-2}

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k	k	k
c	b	a
a	c	b
b	a	c
x_1	x_1	x_1
$\vdots$	$\vdots$	$\vdots$
x_{m-2}	x_{m-2}	x_{m-2}

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- Pareto $\Rightarrow f(P) \subseteq \{a,b,c\}$
- WLOG assume $a \in f(P)$
- First, permute voters
- ρ: 1st k $\longrightarrow$ last k $\longrightarrow$ **mid k**
- $f(\rho P) = f(P)$

k	k	k
c	b	a
a	c	b
b	a	c
x_1	x_1	x_1
⋮	⋮	⋮
x_{m-2}	x_{m-2}	x_{m-2}

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- WLOG assume a $\in$ f(P)
- First, permute voters
- ρ: 1st k $\rightarrow$ last k $\rightarrow$ **mid k**
- f(ρP) = f(P)
- Next, permute alt's

k	k	k
c	b	a
a	c	b
b	a	c
x_1	x_1	x_1
$\vdots$	$\vdots$	$\vdots$
x_{m-2}	x_{m-2}	x_{m-2}

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- First, permute voters
- ρ: 1st k $\rightarrow$ last k $\rightarrow$ **mid k**
- $f(\rho P) = f(P)$
- Next, permute alt's
- with σ: $c \rightarrow a \rightarrow b \rightarrow c$

k	k	k
c	b	a
a	c	b
b	a	c
x_1	x_1	x_1
$\vdots$	$\vdots$	$\vdots$
x_{m-2}	x_{m-2}	x_{m-2}

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- Next, permute alt's
- with σ: c $\rightarrow$ a $\rightarrow$ b $\rightarrow$ c

k	k	k
c a	b c	a b
a b	c a	b c
b c	a b	c a
x_1	x_1	x_1
⋮	⋮	⋮
x_{m-2}	x_{m-2}	x_{m-2}

We'll show a 3-way tie is forced

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- Pareto $\Rightarrow$ f(P) $\subseteq$ {a,b,c}
- WLOG assume a $\in$ f(P)
- First, permute voters
- ρ: 1st k $\rightarrow$ last k $\rightarrow$ **mid k**
- $f(\rho P) = f(P)$
- Next, permute alt's
- with σ: c $\rightarrow$ a $\rightarrow$ b $\rightarrow$ c
- $f(P) = f((\rho P)^{\sigma})$

k	k	k
c a	b c	a b
a b	c a	b c
b c	a b	c a
x_1	x_1	x_1
⋮	⋮	⋮
x_{m-2}	x_{m-2}	x_{m-2}

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- WLOG assume a $\in$ f(P)
- First, permute voters
- ρ: 1st k $\rightarrow$ last k $\rightarrow$ **mid k**
- $f(\rho P) = f(P)$
- Next, permute alt's
- with σ: c $\rightarrow$ a $\rightarrow$ b $\rightarrow$ c
- $f(P) = f((\rho P)^{\sigma})$
  $= \sigma^{-1}[f(\rho P)]$

k	k	k
c a	b c	a b
a b	c a	b c
b c	a b	c a
x_1	x_1	x_1
⋮	⋮	⋮
x_{m-2}	x_{m-2}	x_{m-2}

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- Pareto $\Rightarrow$ f(P) $\subseteq$ {a,b,c}
- WLOG assume a $\in$ f(P)
- First, permute voters
- ρ: 1st k $\to$ last k $\to$ **mid k**
- f(ρP) = f(P)
- Next, permute alt's
- with σ: c $\to$ a $\to$ b $\to$ c
- f(P) = f((ρP)$^{\sigma}$)
  = σ^{-1}[f(ρP)] = σ^{-1}[f(P)] . . .

k	k	k
c a	b c	a b
a b	c a	b c
b c	a b	c a
x_1	x_1	x_1
⋮	⋮	⋮
x_{m-2}	x_{m-2}	x_{m-2}

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- Pareto $\Rightarrow f(P) \subseteq \{a,b,c\}$
- WLOG assume $a \in f(P)$
- First, permute voters
- ρ: 1st k $\rightarrow$ last k $\rightarrow$ **mid k**
- $f(\rho P) = f(P)$
- Next, permute alt's
- with σ: $c \rightarrow a \rightarrow b \rightarrow c$
- $f(P) = f((\rho P)^{\sigma})$
  $= \sigma^{-1}[f(\rho P)] = \sigma^{-1}[f(P)] \ldots$
- So $f(P)$ is closed under σ^{-1}

k	k	k
c a	b c	a b
a b	c a	b c
b c	a b	c a
x_1	x_1	x_1
⋮	⋮	⋮
x_{m-2}	x_{m-2}	x_{m-2}

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- Pareto $\Rightarrow f(P) \subseteq \{a,b,c\}$
- WLOG assume $a \in f(P)$
- First, permute voters
- ρ: 1st k $\rightarrow$ last k $\rightarrow$ **mid k**
- $f(\rho P) = f(P)$
- Next, permute alt's
- with $\sigma: c \rightarrow a \rightarrow b \rightarrow c$
- $f(P) = f((\rho P)^{\sigma})$
  $= \sigma^{-1}[f(\rho P)] = \sigma^{-1}[f(P)]$ . . .
- So $f(P)$ is closed under σ^{-1}, so $\{a,b,c\} \subseteq f(P)$

k	k	k
c a	b c	a b
a b	c a	b c
b c	a b	c a
x_1	x_1	x_1
⋮	⋮	⋮
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We'll show a 3-way tie is forced

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k-approval:
w = (1, ..., 1, 1, 0, ..., 0, 0) with k 1s

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1. Can more than one alternative a have SS(a) > 0?
2. Suppose SS(a) > 0 . . . what can you say about alt. a?

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Why is Top Cycle a Condorcet Extension?

Exercises

- This section contains precise versions of problems mentioned on slides
- Only do the ones you find interesting (there are too many for you to do all right now)
- Most of the tutorial is based on Chapter 2 of the ***Handbook of Computational Social Choice***, Cambridge University Press, 2016. You may find the chapter helpful for these problems.
- Free PDF of the book at http://www.cambridge.org/download_file/898428
- To open the PDF use password: cam1CSC

Exercises

1) Copeland scoring
   - Recall ***symmetric Copeland score*** is given by $Cop(x) = |\{y|x >^{\mu} y\}| - |\{y|y >^{\mu} x\}|$
   - ***Asymmetric Copeland score*** is given by $Cop^{Ass.}(x) = |\{y|x >^{\mu} y\}|$
   - ***Asymmetric+ Copeland score*** is given by $Cop^{Ass.+}(x) = |\{y|x >^{\mu} y\}| + (½)|\{y|y =^{\mu} x\}|$ *

Are these three rules all the same? All different? Answer as completely as possible.

******We write $y =^{\mu} x$ when $Net_P(x>y) = 0$. You will need to consider profiles for an even number of voters, making $y =^{\mu} x$ possible.*

Exercises

2) Scoring weights and affine equivalence

- Scoring vectors $w = w_1, \ldots, w_m$ and $v = v_1, \ldots, v_m$ are ***affinely equivalent*** if there exist constants γ, δ with $\gamma > 0$ such that $v_i = \gamma w_i + \delta$ for each i.
- Prove that two scoring vectors w, v induce the same scoring rule iff they are affinely equivalent.
- Prove that any two evenly spaced vectors are affinely equivalent.
- Prove that ***symmetric*** Borda weights m-1, m-3, . . ., –m+1 yield a total score of β(x) for each alternative x.

Recall that $\beta(x) = \boldsymbol{\Sigma}_{y \in A} Net_P(x>y)$

Exercises

3) Reversal Manipulation We saw Copeland can be ***manipulated via reversal***: a profile P exists for which some voter i can, by completely reversing her ranking, switch the winning alternative from x to some alternative y whom she sincerely prefers (she ranked y over x before reversing)

- Prove that Borda cannot be manipulated via reversal (the same argument shows all scoring rules are similarly immune)
- Prove that Simpson-Kramer can be manipulated via reversal
- **Difficult:** Prove that every resolute Condorcet extension for 4 or more alternatives can be manipulated via reversal