Fair Division of Indivisible Items

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July 2016

COST Summer School | Computational Social Choice

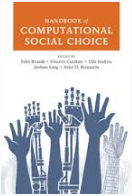
The movie... or is it on Wednesday?

Presentation of this lecture

The course is divided in two parts:

- fairness notions, some properties, computation (Monday)
- protocols for fair allocations (Wednesday)

The content is heavily based on



S. Bouveret, Y. Chevaleyre, N. Maudet. *Fair Division of Indivisible Items.* Handbook of Computational Social Choice.

Presentation of this lecture

- Jupyter Notebook accompanying the lecture.
- Code available at:

  https://github.com/nmaudet/fairdiv-indivisible-items

Other useful resources

- Other general surveys:

Lang and Rothe. *Fair Division of Indivisible Goods.* In Economics and Computation, 2016.

Chevaleyre *et al.* *Issues in multiagent resource allocation.* Informatica, 2006.

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Lang and Rothe. *Fair Division of Indivisible Goods.* In Economics and Computation, 2016.

Chevaleyre *et al. Issues in multiagent resource allocation.* Informatica, 2006.

- Web application

http://www.spliddit.org/

Goldman and Procaccia. *Spliddit: Unleashing Fair Division Algorithms.* SIGecom Exchanges, 2014.

basic notions

Agents and Resources

- a set of agents $\mathcal{N} = \{1, \ldots, n\}$
- a set of resources $\mathcal{O} = \{1, \ldots, m\}$
- an allocation is a function $\pi : \mathcal{N} \rightarrow 2^{\mathcal{O}}$ mapping each agent to the bundle she receive. [$\pi(i)$: bundle/share of agent i]. Set of all allocations Π

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We assume in this lecture that resources:

- cannot be divided,
- cannot be copied, ie. $\pi(i) \cap \pi(j) = \emptyset$, for all $i, j \in \mathcal{N}$

Unless stated otherwise, we also rule out the possibility of special divisible resource (money).

Preferences

Preferences can be expressed ordinal or cardinal.

- ordinal preferences: agents express pre-orders $\succeq$ on $2^{\mathcal{O}}$
- cardinal preferences: agents express utilities $u_i : 2^{\mathcal{O}} \to \mathbb{R}$

One difficulty with resource allocation is that agents potentially have to express over bundles of resources.
(And there is an exponential number of them).

Preferences

Compact representation languages allow to exploit some structures of preferences to get more concise representation, but it depends on the context. For instance:

- if agents only value a few bundles then the bundle form (only expressing non-null values) can be suited,
- if there only limited synergy among resources, then k-additive (only allowing to exress synergies among k resources) utilities can be suited.

Preferences

Another approach is to start from simple preferential information (together with some assumptions, eg. monotonicity) and work with sets of compatible preferences. Eg:

- from a ranking over items, and lift to a ranking over bundles,

In that case the notions can be declined as:

- possible : for some compatible preferences
- necessary : for all compatible preferences

Barbera *et al*. *Ranking sets of objects*. Handbook of utility theory. 2004.

Preferences

In this lecture we shall mainly consider preferences that are:

- additive: the utility enjoyed for a bundle is simply the sum

$$u_i(B) = \sum_{r \in B} u_i(\{r\})$$

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$$u_i(\mathcal{O}) = K, \forall i \in \mathcal{N}$$

- Borda: agents (strictly) rank resources and assign utility m to their preferred resource, $m-1$ to their second preferred, etc.

$$r_0 \succ r_3 \succ r_1 \succ r_2 \Rightarrow \begin{array}{cccc} r_0 & r_1 & r_2 & r_3 \\ 4 & 2 & 1 & 3 \end{array}$$

Preferences

When we perform experiments in particular, it is important to specify how they will preferences will be drawn:

- uniform: the utility of each item is drawn from uniform distribution in a given interval.
- correlated: for each item r an intrinsic utility $u^*(r)$ is drawn. Then each agent's utility is draw with normal distribution centered on $u^*(r)$.

efficiency

Efficiency

Although we are mainly concerned with fairness notions, efficiency requirements are still important, otherwise we may promote rather radical solutions:

Throw these candies if you can't decide how to split it.

Efficiency

- completeness: all resources must be allocated:

$$\cup_{i \in \mathcal{N}} \pi(i) = \mathcal{O}$$

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Efficiency

- completeness: all resources must be allocated:

$$\cup_{i \in \mathcal{N}} \pi(i) = \mathcal{O}$$

- Pareto-optimality: no other allocation is as good for everyone (and strictly better for at least one agent)
- utilitarian social welfare: maximize the sum of agents' utilities

$$\sum_{i \in \mathcal{N}} u_i(\pi(i))$$

[Observe that completeness only refers to allocation, Pareto-optimality is ordinal, utilitarian social welfare is cardinal]

fairness measures and criteria

Measures and Criteria

Different ways to assess how fair is an allocation have been proposed in the literature.

- some of them are measures, which can be optimized,
- others are boolean criteria, which are satisfied or not.

Egalitarian social welfare

The egalitarian social welfare tries to maximize the utility of the agent who is the worst-off.

$$\min_{i \in \mathcal{N}} u_i(\pi(i))$$

An allocation maximizing this value egalitarian-optimal.

Nash social welfare

The Nash social welfare tries to maximize the product of utilities of the agents.

$$\prod_{i \in \mathcal{N}} u_i(\pi(i))$$

An allocation maximizing this value is Nash-optimal.

Proportionality

The proportional fair share of an agent is the n^{th} of the utility she assigns to the full bundle

$$pfs(i) = \frac{1}{n} u_i(\mathcal{O})$$

An allocation π is proportional if $u_i(\pi(i)) \geq pfs(i)$, for all $i \in \mathcal{N}$

Steinhaus. *The problem of fair division.* Econometrica. 1948.

Maxmin Fair Share

The maxmin fair share of an agent is the best share she can guarantee herself in a game "I cut, I choose last".

$$mfs(i) = \max_{\pi \in \Pi} \min_{j \in \mathcal{N}} u_i(\pi(j))$$

An allocation π satisfies maxmin fair share if $u_i(\pi(i)) \geq mfs(i)$, for all $i \in \mathcal{N}$

Budish. *The combinatorial assignment problem.* Journal of Political Economy. 2011.

Envy-freeness

An allocation is envy-free when no agent prefers the bundle of another agent over her own bundle.

$$u_i(\pi(i)) \geq u_i(\pi(j)), \text{for all } \ i, j \in \mathcal{N}$$

Foley. *Resource allocation and the public sector.* Yale Econ Essays. 1967.

Degrees of Envy

It is possible to define various notions of degrees of envy, by combining operators at different levels:

- envy between two agents : we may consider

$$e_{ij} = max(u_i(\pi(j)) - u_i(\pi(i)), 0)$$

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- degree of envy of a single agent i: $\max_{j \in \mathcal{N}} e_{ij}$, or $\sum_{j \in \mathcal{N}} e_{ij}$
- degree of envy of the society: $\max_{i \in \mathcal{N}} e_i$, or $\sum_{i \in \mathcal{N}} e_i$

Y. Chevaleyre, U. Endriss, N. Maudet. *Distributed Fair Allocation of Indivisible Goods.* Working paper.

Degrees of Envy

Essentially, from the matrix of envies $M_e(\pi)$ for an allocation π, you can derive many envy measures $e(\pi)$.

Intuitive axiom to satisfy: $e(\pi) = 0$ iff π is indeed envy-free.

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Many measures can be found in the literature: For instance

"minimize the maximum envy between any pair of agents"

is proposed by (Lipton *et al.*), this corresponds to $e^{max,max,raw}$.

Lipton *et al.* *On approximately fair allocations of indivisible goods.* EC-04.

Envy-Freeness up to one good

The envy of *i* towards *j* can be eliminated by removing a single resource from *j*'s bundle.

$$\forall i, j \in \mathcal{N}\ \exists r \in \pi(j) : u_i(\pi(i)) \geq u_i(\pi(j) \setminus \{r\})$$

A stronger version requiring that the removal of any resource eliminates envy ("envy up to the least envied good"):

$$\forall i, j \in \mathcal{N}\ \forall r \in \pi(j) : u_i(\pi(i)) \geq u_i(\pi(j) \setminus \{r\})$$

Caragiannis *et al.*. *The Unreasonable Fairness of Maximum Nash Welfare.* EC-16.

Fairness Measures and Criteria: Recap

To sum up, we shall mainly use:

- (PROP) proportionality
- (MFS) maxmin fair share
- (EF) envy-freeness
- (ESW) egalitarian social welfare
- ($e^{sum,max,bool}$) number of envious agents
- ($e^{max,max,raw}$) max envy between any pair of agents
- envy up to one good

Example

Two agents, five resources, additive utilities.

	r_1	r_2	r_3	r_4	r_5
agent 1	6	6	6	0	0
agent 2	5	5	3	3	2

Example

Two agents, five resources, additive utilities.

	r_1	r_2	r_3	r_4	r_5	*pfs*	*mfs*
agent 1	6	6	6	0	0	$18/2 = 9$	6
agent 2	5	5	3	3	2	$18/2 = 9$	8

Example

Two agents, five resources, additive utilities.

	r_1	r_2	r_3	r_4	r_5	pfs	mfs
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- what is the egalitarian social welfare for this problem?

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- find an allocation satisfying MFS but not PROP
- does there exist an EF allocation?
- what is the egalitarian social welfare for this problem?
- is there an allocation such that both agents would be envious?

some results (quizz)

Some results: true or false?

☞ *If $m < n$, none of the criteria can be satisfied*

Some results: true or false?

☞ *If $m < n$, none of the criteria can be satisfied*

Almost true, but false.
(This is actually true for all our criteria, as one agent is bound to be left without any resource. However, note that the mfs of all agents in that case will be 0, hence MFS is trivially satisfied.)

Some results: true or false?

☞ *With additive preferences, an EF allocation is necessarily Pareto-optimal*

Some results: true or false?

☞ *With additive preferences, an EF allocation is necessarily Pareto-optimal*

False. Here is a counter-example:

	r_1	r_2	r_3	r_4
agent 1	$\boxed{10}$	0	$\boxed{1}$	2
agent 2	0	$\boxed{10}$	2	$\boxed{1}$

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	r_1	r_2	r_3	r_4
agent 1	$\boxed{10}$	0	1	$\boxed{2}$
agent 2	0	$\boxed{10}$	$\boxed{2}$	1

Some results: true or false?

☞ *At least an egalitarian-optimal must be Pareto-optimal*

Some results: true or false?

☞ *At least an egalitarian-optimal must be Pareto-optimal*

True.
Suppose by contradiction that this is not the case: any egalitarian-optimal is Pareto-dominated by π' not egalitarian-optimal. But either:

- $min(\pi) = min(\pi')$, but then π' is egalitarian-optimal.
- $min(\pi) < min(\pi')$, but then π is not egalitarian-optimal.

Contradiction.

Some results: true or false?

☞ *With additive preferences, an EF allocation is proportional*

Some results: true or false?

☞ *With additive preferences, an EF allocation is proportional*

Suppose for the sake of contradiction that this is not the case.

- We have an allocation EF and not PROP, hence it must be that there is an agent i s.t. $u_i(\pi(i)) < \frac{1}{n} u_i(\mathcal{O})$
- thus $u_i(\mathcal{O} \setminus \pi(i)) > \frac{(n-1)}{n} . u_i(\mathcal{O})$
- but then at least one of the $(n-1)$ agents must hold a bundle that agent i values more than $\frac{1}{n} u_i(\mathcal{O})$. Contradiction.

Some results: true or false?

☞ *For an agent i, it is the case that* $mfs(i) \leq pfs(i)$

Some results: true or false?

☞ *For an agent i, it is the case that $mfs(i) \leq pfs(i)$*

True.
Intuitively, the proportional fair share can be seen as the "ideal case" where an agent could split all resources as she wished, so as to guarantee that all shares have the same value for her.

Some results: true or false?

☞ *With additive utilities, an MFS allocation always exists*

Some results: true or false?

☞ *With additive utilities, an MFS allocation always exists*

False. But very hard to show... In fact :

- it is possible to construct instances with $m = n^n$ items not satisfying MFS
- true for many special cases, almost always verified in practice
- possible to guarantee for all agents 2/3 of their mfs

Procaccia and Wang. *Fair enough: Guaranteeing approximate maximin shares.* EC-14.

Bouveret and Lemaître. *Characterizing Conflicts in Fair Division of Indivisible Goods Using a Scale of Criteria.* JAAMAS-2014.

Amanatidis *et al.* *Approximation Algorithms for Computing Maximin Share Allocations.* ICALP-15.

Some results

☞ *With Borda utilities, any EF allocation is ESW-optimal*

A scale of fairness measures

For additive preferences, the following scale of criteria holds:

$$(EF) \Rightarrow (PROP) \Rightarrow (MFS)$$

This suggests a possible approach: first ask for the more demanding criteria, then move to the next weaker, etc.

Was actually used in practice in spliddit (+ utilitarian welfare maximization on top).

S. Bouveret and M. Lemaitre. *Characterizing Conflicts in Fair Division of Indivisible Goods Using a Scale of Criteria.* JAAMAS-2014.

Goldman and Procaccia. *Spliddit: Unleashing Fair Division Algorithms.* SIGecom Exchanges, 2014.

… but Nash social welfare may well be compelling

"The Nash solution exhibits an elusive combination of fairness and efficiency properties, and can be easily computed in practice. It provides the most practicable approach to date (arguably, the ultimate solution) for the division of indivisible goods under additive valuations."

Recently deployed on spliddit.

Caragiannis *et al.*. *The Unreasonable Fairness of Maximum Nash Welfare.* EC-16.

The case for Nash social welfare

- guarantees Envy-freeness up to one good and Pareto-optimality
- provides guarantees on an approximation of MFS, and in practice (on spliddit instances) provides full MFS

Efficiency vs. Fairness

As we have seen, there is usually no simple relation between efficiency measures and fairness.

A natural question is thus to ask what is the cost of fairness

$$\frac{\text{utilitarian opt}}{\text{utilitarian sw of best allocation satisfying fairness}}$$

This is known as the price of fairness.

Caragiannis. *Fairness and Efficiency*. COST Summer School.

Caragiannis *et al*. *The efficiency of fair division*. TOCS-2012.

Efficiency vs. Fairness

Eg., what is the price to pay if we insist on the allocation to be ESW-optimal?

Consider the following example (with additive utilities).

	r_1	r_2	r_3	r_4	r_5
agent 1	ϵ	$1-\epsilon$	0	0	0
agent 2	0	ϵ	$1-\epsilon$	0	0
agent 3	0	0	ϵ	$1-\epsilon$	0
agent 4	0	0	0	ϵ	$1-\epsilon$
agent 5	0	0	0	0	1

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agent 4	0	0	0	ϵ	$1-\epsilon$
agent 5	0	0	0	0	1

$$\Rightarrow \quad \text{PoF} \quad = \frac{(n-1)(1-\epsilon)}{1+(n-1)\cdot\epsilon} = n$$

Caragiannis *et al. The efficiency of fair division.* TOCS-2012.

solving the problem

We first address the question of the computation, by a central authority, of the notion mentioned earlier.
In particular, we will have a look at :

- computing ESW-optimal (ie. maxmin) allocation
- computing envy-free allocations

Computing maxmin allocations

In general, the problem is computationally difficult (NP-hard): it is likely that no polynomial algorithm can solve this problem.

What is perhaps more surprising is that the problem remains difficult even if agents have additive utilities.
This problem has been studied as the Santa Claus problem.

[By comparison, observe that computing an utilitarian-optimal allocation is easy in that case.]

Bansal and Sviridenko. *The Santa Claus problem.* STOC-2006.

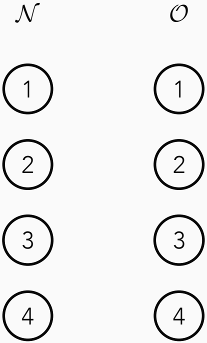
The assignment MIP

Let us make use of binary variables $x_i j$ to express that agent i holds resource r_j (=1), or not (=0)

$$
\begin{array}{lll}
\text{maximize} & y & \\
\text{subject to} & x_{i,j} \in \{0,1\}, & j = 1, \ldots, m \\
 & & i = 1, \ldots, n \\
 & \sum_{i \in N} x_{ij} = 1, & j = 1, \ldots, m \\
 & \sum_{r_j \in O} u_i(o_j) \times x_{ij} \geq y, & i = 1, \ldots, n
\end{array}
$$

Example of a tractable case: $m = n$

For illustration, suppose preferences are Borda.



agent 1	$r_1 \succ$	$r_2 \succ$	$r_3 \succ$	r_4
agent 2	$r_2 \succ$	$r_1 \succ$	$r_4 \succ$	r_3
agent 3	$r_1 \succ$	$r_2 \succ$	$r_3 \succ$	r_4
agent 4	$r_3 \succ$	$r_4 \succ$	$r_1 \succ$	r_2

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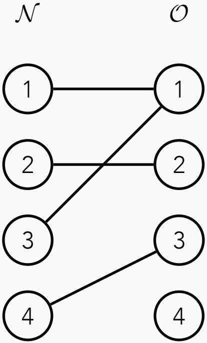

	$\downarrow$			
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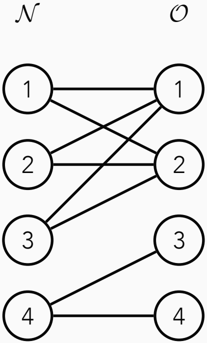

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agent 1	$r_1 \succ$	$r_2 \succ$	$r_3 \succ$	r_4
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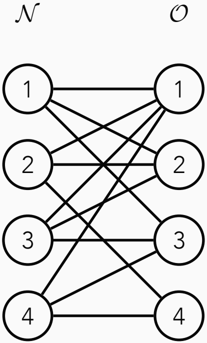

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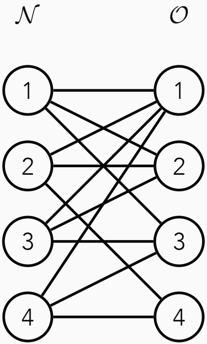

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As perfect matching runs in $O(n^3)$, this algorithm runs in $O(n^4)$.
Matching techniques handle more general maxmin problems.

Golovin. *Maxmin fair allocation of indivisible goods.* Tech. report.

Envy

Computing envy-free allocation is...

Envy

Computing envy-free allocation is...
simple if we don't impose any efficiency requirement.

Even with additive utilities, deciding whether there exists:

- a complete envy-free allocation is NP-complete,
- a Pareto-optimal envy-free allocation is even higher in the hierarchy.

Lipton *et al.* *On approximately fair allocations of indivisible goods.* EC-04.

de Keijzer. *On the complexity of efficiency and envy-freeness in fair division of indivisible goods with additive preferences.* ADT-09.

The envy-minimizing MIP

We can express

$$
\begin{array}{lll}
\text{minimize} & y & \\
\text{subject to} & \quad x_{i,j} \in \{0,1\}, & j = 1,\ldots,m \\
& & i = 1,\ldots,n \\
& \displaystyle\sum_{i \in N} x_{ij} = 1, & j = 1,\ldots,m \\
& \displaystyle\sum_{r_j \in O} u_i(o_j) \times x_{i'j} - \sum_{r_j \in O} u_i(o_j) \times x_{ij} \leq y, & i, i' = 1,\ldots,n \\
& & i' \neq i'
\end{array}
$$

Many more results...

- Similar results have been obtained for many more fairness measures.

Furthermore, to circumvent the difficulty of many of these problems, it is possible to study:

- approximation algorithms, which would return solution with worst-case guarantees wrt. optimal solutions,
- study specific cases, in terms of preference structures, number of agents, number of resources, etc.

Nguyen, Roos, Rothe. *A survey of approximability and inapproximability results for social welfare optimization in multiagent resource allocation.* AMAI13.

How likely is it to get an EF allocation?

Intuitively, the more resources we get, the easier it should be to get an EF allocation.

Can we get a more precise statement about that?

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Can we get a more precise statement about that?

(Dickerson *et al.*) studied this for uniform and correlated utilities (in fact for more general distribution satisfying some axioms), getting asymptotic results and experimental evidence.

- the number of items needed on top of n to ensure envy-free is linear in n
- a phase transition phenomena is observed

Dickerson *et al.* *The computational rise and fall of fairness.* AAAI-14.