Introduction to the theory and practice of stable matchings

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The aim of this talk

- overview of various markets
- the basic model of Gale-Shapley and their algorithm
- examples to explain the notions
- practically motivated extensions of the basic model
- I will provide some proofs
- exercises to get hand-on various models and algorithms
- Exercises extend the material by:
  - non-bipartite version
  - alternative stability definition: exchange stability
- Necessarily many topics left out

How it all started

D. Gale: The two-sided matching problems. Origin, development and current issues, Int. Game Theory Review 3 (2001) 237-252.

...it all started with an article in the New Yorker magazine, 10 September 1960, in which a reporter spent several weeks observing the operation of the undergraduate admission office of Yale University. Early in the article, the reporter observes,

> " the admissions men very often have no way of discovering how many other colleges each applicant is trying for, nor have they any way of knowing how many students they decide to admit actually intend to come to their college..."

with the consequence that the admissions officer may end up "discovering that he has acceptances from a freshman class either half as large or twice as large as the school has room for.“

...because of all the guess work, one would expect the final allocation of applicants to colleges would be highly "non-optimal", so the first problem was to pin down precisely the nature of these "non-optimalities". With this in mind, I decided to look first at the special case where each college has a quota of one. This is, of course, highly unnatural for the college problem, so

> for the sake of local color the scenario was changed.

Recent statistics in Slovakia

(Institute of information and forecasting in education)

School (compared to plan)	applications	admitted	started
Medical faculty Bratislava	3.78	0.87	0.75
Medical faculty Martin	4.96	0.61	0.46
Medical faculty Košice	3.55	1.18	0.99
Pedagogical faculty Bratislava	0.86	0.54	0.31
Pedagogical faculty Ružomberok	0.71	0.62	0.49
Science faculty Košice	1.17	0.77	0.33
- mathematics (plus economics) – plan 80	0.97	0.50	0.10

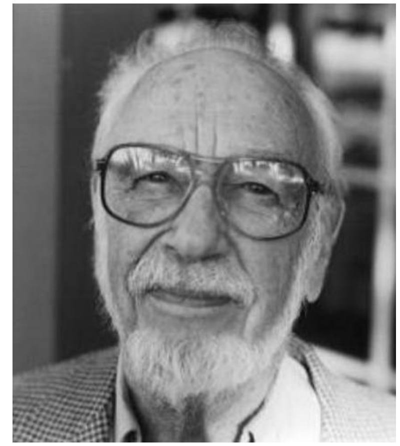
multiplicity of applications	1	2	3	4	5	...	10	11	12
No. of persons	18 130	10 925	5 430	1 983	611		3	1	1

- National Residents Matching Program (USA)

Roth, A.E., "The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory," Journal of Political Economy, 92, 1984, 991-1016

- internship: a form of postgraduate medical education since ~1900
- for hospitals: a supply of relatively cheap labor → competition among hospitals for interns
- hospitals: tried to set the date for the binding agreements earlier than their competitors
- 1944: date of appointment 2 full years before the internship was actually to begin.
- students waited for offers from preferable position, hospitals got last minutes rejections
- centralized clearinghouse: instead of hospitals making individual offers and students respond, students and programs submit rank order list to indicate their preferences
- 1950-1951 trial run of a centralized algorithm, 1951-1952 new NIMP algorithm
- very high levels of voluntary paticipation up to present, key feature: **stability**

- 2012 Nobel Memorial Prize in Economics



Roth, Alvin E. Deferred Acceptance Algorithms: History, Theory, Practice, and Open Questions, International Journal of Game Theory, Special Issue in Honor of David Gale on his 85th birthday, 36, March, 2008, 537-569.

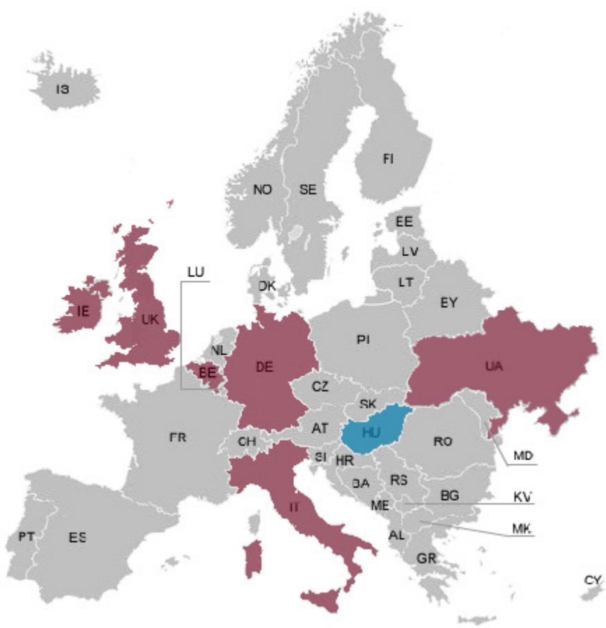
- deferred accepance algorithms have been independently developed in various markets (> 50)
- changes to accommodate various requirements of the market**: married couples**
- 2002: law firms brought an antitrust suit against the matching system (conspiracy to hold down wages for residents
- the use of deferred acceptance algorithm has been explicitly recongnized as part of pro-competitive market mechanism in American law

Matching in Practice

European network for research on matching practices in education and related markets

Home | About | People | Events | Matching Practices in Europe | Research

Countries or regions with available information are coloured burgundy. For more information please click on each country or region.



Organization of Education	Mostly public universities, some run by Churches and a small number of private universities.
Stated Objectives	Freedom of choice. Free studies for the best students. Financial contribution is possible for admission.
Who's in Charge?	A non-profit governmental organisation.
In Place Since	1985
Available Capacity	Decided jointly by the universities and the Ministry (click on the country for more info).
Timing of Enrolment	Three matching rounds: the main one terminates in July 2nd one in August. The 3rd run starts in spring for MSc programmes.
Information	All the revelant information is available through centrally published booklets and website.
Restrictions on Preference Expression	No restriction, but the applicants are charged for every item in their lists after the third one.
Matching Procedure	A score-limit algorithm based on the student-proposing Gale-Shapley algorithm. (Click the country for more info)
Priorities and Quotas	Priorities based on the scores of students. (Click the country for more info)
Tie-breaking	No tie-breaking, students with equal scores are either rejected or accepted together ('equal treatment policy').
Further Special Feature	Students may apply for a pair of programmes in the case of teaching studies, the nature is same

Higher education systems based on scores of students (Biró et al.)

- Hungary, Ireland, Spain Turkey
- students scores based on grades and entrance exams
- the score of a student for different schools may be different
- score limit: the lowest score that allows students to be admitted
- each student is admitted to the first place on her list where she achieved the score-limit
- Rules for breaking ties in case of equal scores:
  - date of birth (Turkey)
  - lotery (New York, Boston)
  - so as to maximize the size of matching (Scottish Foundation Allocation Scheme)
  - equal treatment policy (Hungary)

Applications today

- National Residents Matching Program (USA)
- Canadian Resident Matching Service
- Scottish PRHO Allocations scheme
- Admissions to public schools in New York, Boston
- University admissions in Hungary
- Large-scale residence exchange in Chinese housing markets
    - Yuan, 1996
- Allocation of campus housing in American universities, such as Carnegie-Mellon, Rochester and Stanford
    - Abdulkadiroğlu and Sönmez, 1998
- US Naval Academy: students to naval officer positions
    - Roth and Sotomayor, 1990
- Scottish Executive Teacher Induction Scheme
- Assigning students to projects
- Search of donors for kidney transplantations
    - A. Roth, T. Sönmez, U. Ünver (2005)

Not so successful stories

Not so successful stories

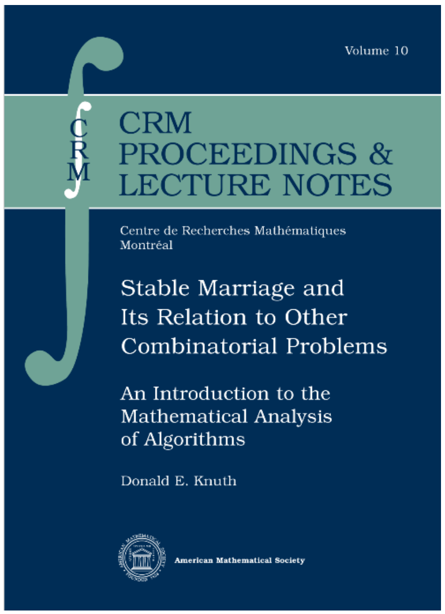
Na Feira, Crato ouviu dois coros: um afinado de alunos, outro de protesto de professores sem trabalho

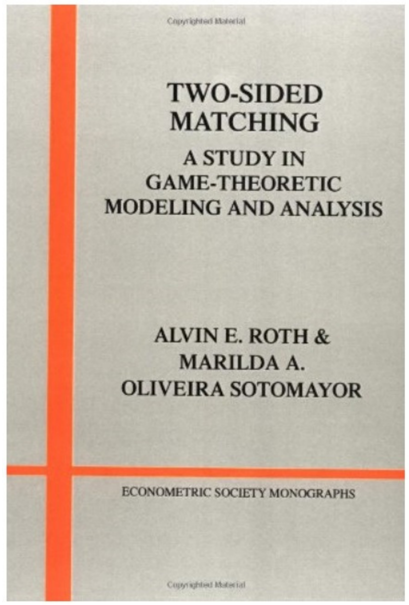
SARA DIAS OLIVEIRA 26/09/2014 - 18:20

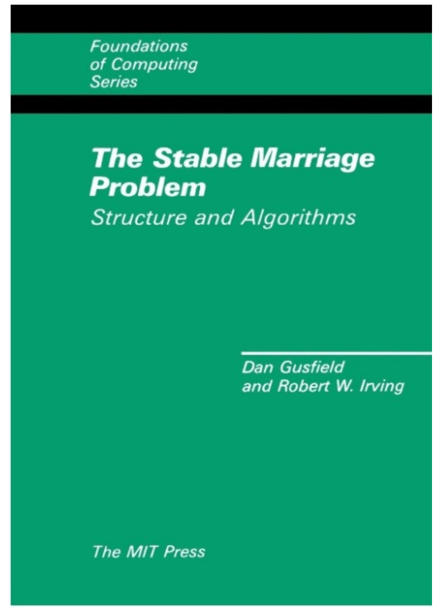
MULTIMÉDIA

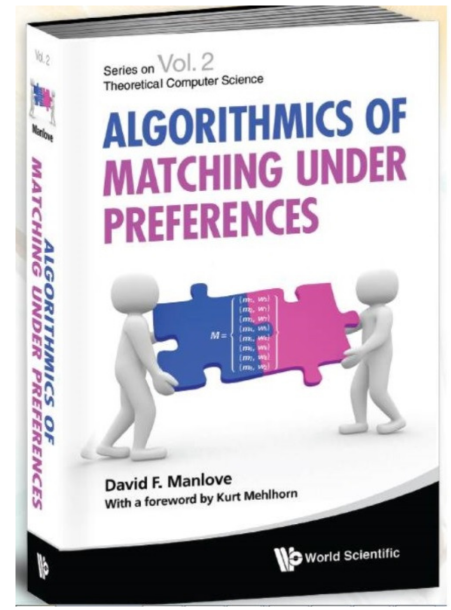
Nesta sexta-feira, o ministro da Educação referiu-se a um "dia de festa" pela inauguração da nova EB2,3 Fernando Pessoa, em Santa Maria da Feira, escola com 1160 alunos, 41 turmas, 90 professores e 31 auxiliares. À sua espera, dentro da escola, estava um coro afinado de alunos do 9.º ano com várias canções preparadas e acompanhadas ao piano. Lá fora, rodeado por um cordão policial, um coro de protestos em alta voz, megafone em punho, mobilizado pelo movimento nacional de professores Boicote e Cerco, com "Crato rua, a escola não é tua" na ponta das línguas e um cartaz com uma fórmula matemática: "Caos nos concursos = alunos sem aulas + 40.000 professores sem trabalho."

Books on the topic









- plus chapters in handbooks (of Game Theory, Computational Social Choice)
- exploding literature

The stable marriage problem

D. Gale and L. S. Shapley, College admissions and the stability of marriage, Amer. Math. Monthly 69 (1962), 9-15.

- a set of men $M=\{m_1,m_2,\dots,m_n\}$
- a set of women $W=\{w_1,w_2,\dots,w_n\}$
- each person has a complete linear ordering of persons of the opposite sex = preference list
- preference profile

  $P= (P(m_1),\dots, P(m_n); P(w_1),\dots, P(w_n))$
- An instance of the Stable marriage problem (SM) is I=(M,W,P).

Example 1.

$P(m_1)$: w_1,w_2,w_3,w_4
$P(m_2)$: w_1,w_4,w_2,w_3
$P(m_3)$: w_2,w_1,w_3,w_4
$P(m_4)$: w_3,w_4,w_2,w_1

$P(w_1)$: m_3,m_4,m_2,m_1
$P(w_2)$: m_3,m_4,m_2,m_1
$P(w_3)$: m_3,m_2,m_1,m_4
$P(w_4)$: m_1,m_3,m_2,m_4

This means:

for man m_1: woman w_1 is his first choice,
woman w_2, is his second choice etc.

We say: man m_1 prefers woman w_1 to woman w_2

We write: $w_1 >_{m_1} w_2$

What we are looking for

Definition 1. A matching μ is a set of disjoint man-women pairs.

Definition 2. A pair (m,w) is a blocking pair for a matching μ if both m and w prefer the other to their current partners in μ.

Definition 3. A matching μ is stable if it does not admit a blocking pair.

Example 1 – basic notions.

$P(m_1)$: w_1, w_2, w_3, w_4

$P(m_2)$: w_1, w_4, w_2, w_3

$P(m_3)$: w_2, w_1, w_3, w_4

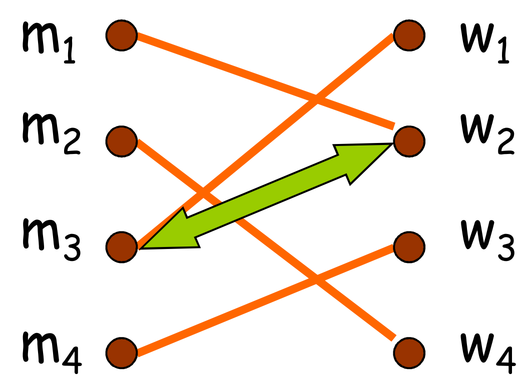
$P(m_4)$: w_3, w_4, w_2, w_1

$P(w_1)$: m_3, m_4, m_2, m_1

$P(w_2)$: m_3, m_4, m_2, m_1

$P(w_3)$: m_3, m_2, m_1, m_4

$P(w_4)$: m_1, m_3, m_2, m_4



Matching μ:

$$\mu = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_2 & w_4 & w_1 & w_3 \end{pmatrix}$$

μ is not stable, as e.g. the pair

(m_3, w_2) is blocking.

Theorem 1 (Gale & Shapley). A stable matching always exists.

Gale-Shapley algorithm - man propose

begin assign each person to be free;
    while some man m is free **do**
    begin w:=first woman to whom m has not yet proposed;
        if w is free
        then assign m and w to be engaged
        else if w prefers m to her fiancé m'
            then assign m and w to be engaged and m' to be free
            else w rejects m
    end
    output the stable matching consisting of engaged pairs
end

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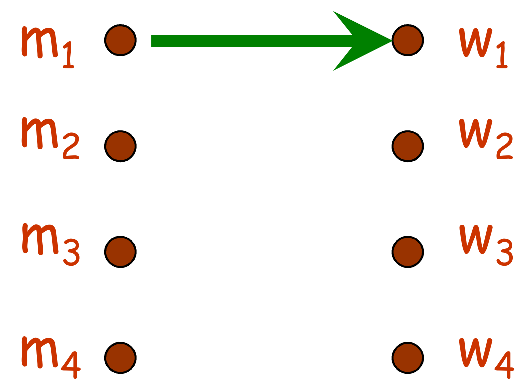
Example 1.Gale-Shapley algorithm

$P(m_1)$: w_1, w_2, w_3, w_4 $P(w_1)$: m_3, m_4, m_2, m_1

$P(m_2)$: w_1, w_4, w_2, w_3 $P(w_2)$: m_3, m_4, m_2, m_1

$P(m_3)$: w_2, w_1, w_3, w_4 $P(w_3)$: m_3, m_2, m_1, m_4

$P(m_4)$: w_3, w_4, w_2, w_1 $P(w_4)$: m_1, m_3, m_2, m_4



Start of the algorithm. All persons are free.

Take m_1: proposes to w_1.

Example 1.Gale-Shapley algorithm

$P(m_1)$: w_1,w_2,w_3,w_4	$P(w_1)$: m_3,m_4,m_2,m_1
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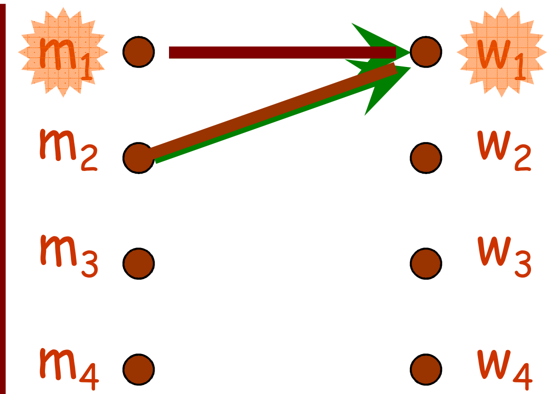
$P(m_3)$: w_2,w_1,w_3,w_4	$P(w_3)$: m_3,m_2,m_1,m_4
$P(m_4)$: w_3,w_4,w_2,w_1	$P(w_4)$: m_1,m_3,m_2,m_4



w_1 is free, so m_1 and w_1 become engaged.

Take m_2: proposes to w_1.

w_1 prefers m_2 to m_1.

So m_2&w_1 become engaged and m_1 is set free.

Example 1.Gale-Shapley algorithm

$P(m_1)$: w_1, w_2, w_3, w_4
$P(m_2)$: w_1, w_4, w_2, w_3
$P(m_3)$: w_2, w_1, w_3, w_4
$P(m_4)$: w_3, w_4, w_2, w_1

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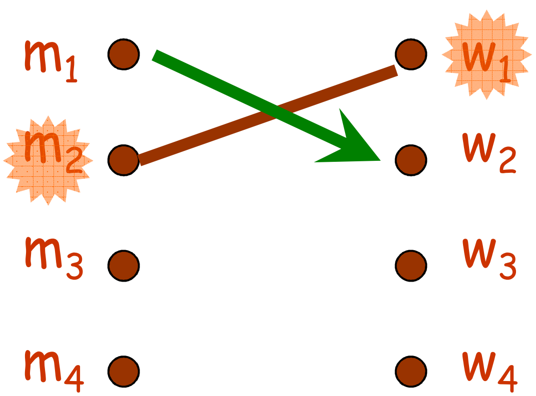


Take m_1: proposes to the first woman, he has not proposed yet: w_2.

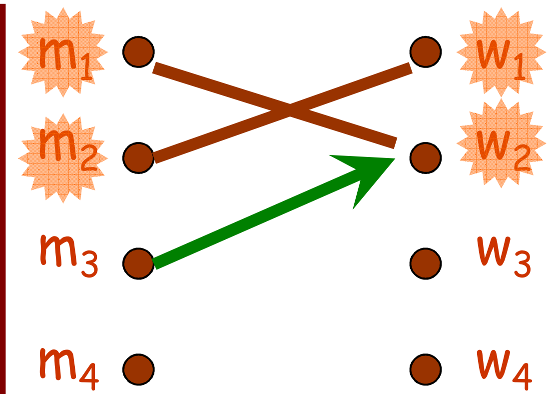
Example 1.Gale-Shapley algorithm

$P(m_1)$: w_1, w_2, w_3, w_4 $P(w_1)$: m_3, m_4, m_2, m_1

$P(m_2)$: w_1, w_4, w_2, w_3 $P(w_2)$: m_3, m_4, m_2, m_1

$P(m_3)$: w_2, w_1, w_3, w_4 $P(w_3)$: m_3, m_2, m_1, m_4

$P(m_4)$: w_3, w_4, w_2, w_1 $P(w_4)$: m_1, m_3, m_2, m_4



Take m_1: proposes to the first women, he has not proposed yet: w_2.

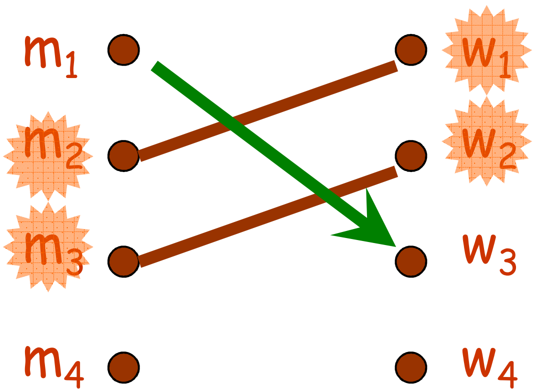
w_2 is free, so m_1 and w_2 become engaged.

Now m_3 proposes to w_2.

w_2 prefers m_3 to m_1. So m_3&w_2 become engaged and m_1 is set free.

Example 1.Gale-Shapley algorithm

$P(m_1)$: w_1, w_2, w_3, w_4	$P(w_1)$: m_3, m_4, m_2, m_1
$P(m_2)$: w_1, w_4, w_2, w_3	$P(w_2)$: m_3, m_4, m_2, m_1
$P(m_3)$: w_2, w_1, w_3, w_4	$P(w_3)$: m_3, m_2, m_1, m_4
$P(m_4)$: w_3, w_4, w_2, w_1	$P(w_4)$: m_1, m_3, m_2, m_4



m_1: proposes to the first woman, he has not proposed yet: w_3.

w_3 is free, so m_1 and w_3 become engaged.

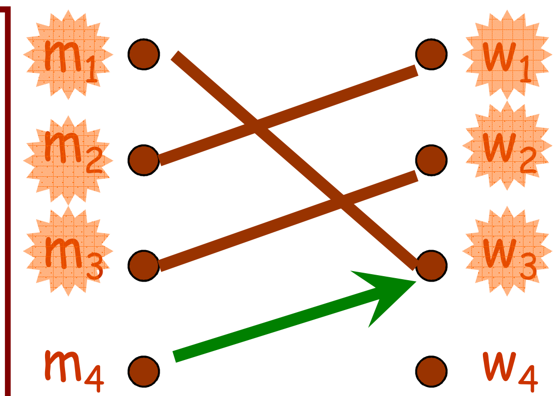
Example 1.Gale-Shapley algorithm

$P(m_1)$: w_1, w_2, w_3, w_4 — $P(w_1)$: m_3, m_4, m_2, m_1

$P(m_2)$: w_1, w_4, w_2, w_3 — $P(w_2)$: m_3, m_4, m_2, m_1

$P(m_3)$: w_2, w_1, w_3, w_4 — $P(w_3)$: m_3, m_2, m_1, m_4

$P(m_4)$: w_3, w_4, w_2, w_1 — $P(w_4)$: m_1, m_3, m_2, m_4



m_1: proposes to the first women, he has not proposed yet: w_3.

w_3 is free, so m_1 and w_3 become engaged.

Now m_4 proposes to w_3. w_3 prefers her fiancé to m_4, so rejects m_4 .

Example 1.Gale-Shapley algorithm

$P(m_1)$: w_1, w_2, w_3, w_4

$P(m_2)$: w_1, w_4, w_2, w_3

$P(m_3)$: w_2, w_1, w_3, w_4

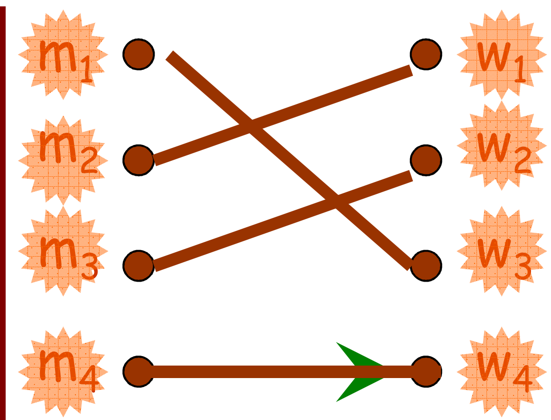
$P(m_4)$: w_3, w_4, w_2, w_1

$P(w_1)$: m_3, m_4, m_2, m_1

$P(w_2)$: m_3, m_4, m_2, m_1

$P(w_3)$: m_3, m_2, m_1, m_4

$P(w_4)$: m_1, m_3, m_2, m_4



m_4: proposes to the first women, he has not proposed yet: w_4.

w_4 is free, so m_4 and w_4 become engaged.

Final matching: $\mu = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_3 & w_1 & w_2 & w_4 \end{pmatrix}$

Theorem 1. For any instance of the stable marriage problem, the Gale-Shapley algorithm terminates, and, on termination, the engaged pairs constitute a stable matching.

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Proof. Let (m,w) be a blocking pair for μ

- This means:
  - man m prefers woman w to his wife w′ in μ and
  - woman w prefers man m to her husband m′ in μ

P(m): … w … w′…

- hence during Gale-Shapley m proposed to w before he proposed to his wife, but was rejected
- why? because w got a proposal from a more preferred man
- so w is married in μ to a man m′ she prefers to m

P(w): … m′… m …

- so (m,w) is not a blocking pair after all

Example 1.Gale-Shapley algorithm with women proposing

$P(m_1)$: w_1, w_2, w_3, w_4	$P(w_1)$: m_3, m_4, m_2, m_1
$P(m_2)$: w_1, w_4, w_2, w_3	$P(w_2)$: m_3, m_4, m_2, m_1
$P(m_3)$: w_2, w_1, w_3, w_4	$P(w_3)$: m_3, m_2, m_1, m_4
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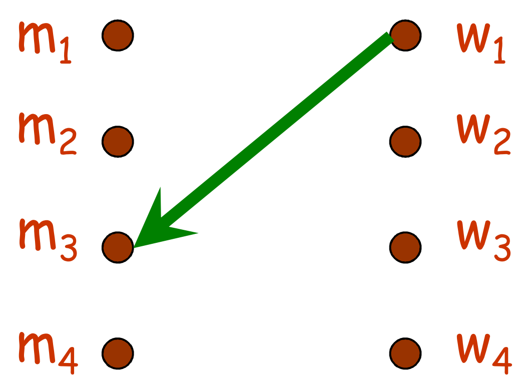


Start of the algorithm. All persons are free.

Take w_1: proposes to m_1 and (m_1, w_1) become engaged.

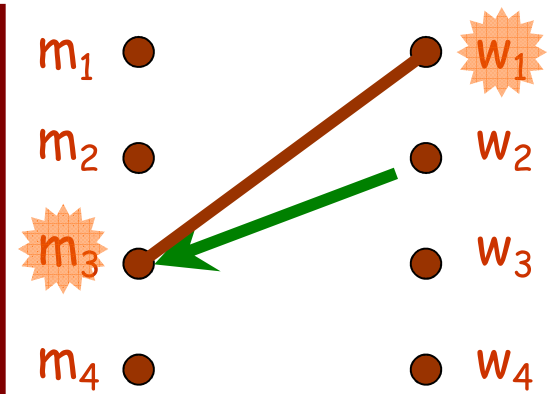
Example 1.Gale-Shapley algorithm with women proposing

$P(m_1)$: w_1, w_2, w_3, w_4 $P(w_1)$: m_3, m_4, m_2, m_1

$P(m_2)$: w_1, w_4, w_2, w_3 $P(w_2)$: m_3, m_4, m_2, m_1

$P(m_3)$: w_2, w_1, w_3, w_4 $P(w_3)$: m_3, m_2, m_1, m_4

$P(m_4)$: w_3, w_4, w_2, w_1 $P(w_4)$: m_1, m_3, m_2, m_4



Now w_2 proposes to m_3

m_3 prefers w_2 to w_1, so m_3&w_2 become engaged and w_1 is set free.

Example 1.Gale-Shapley algorithm with women proposing

$P(m_1)$: w_1, w_2, w_3, w_4	$P(w_1)$: m_3, m_4, m_2, m_1
$P(m_2)$: w_1, w_4, w_2, w_3	$P(w_2)$: m_3, m_4, m_2, m_1
$P(m_3)$: w_2, w_1, w_3, w_4	$P(w_3)$: m_3, m_2, m_1, m_4
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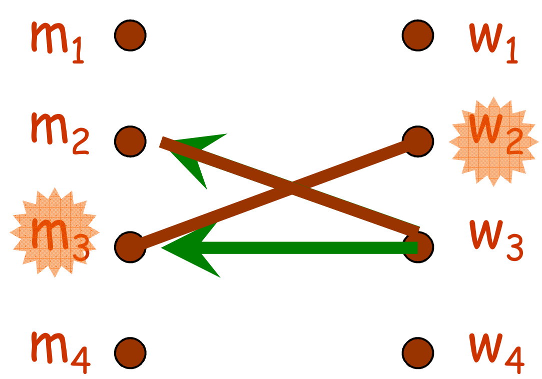
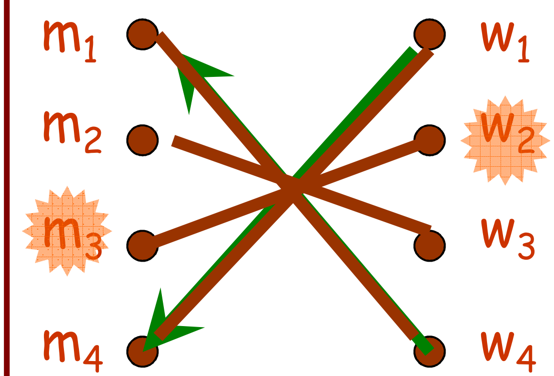


Now w_3 proposes to m_3 m_3 prefers her fiancé to w_3 so rejects w_3 .

Now w_3 proposes to m_2 m_2 is free, so m_2 and w_3 get engaged.

Example 1.Gale-Shapley algorithm with women proposing

$P(m_1)$: w_1,w_2,w_3,w_4	$P(w_1)$: m_3,m_4,m_2,m_1
$P(m_2)$: w_1,w_4,w_2,w_3	$P(w_2)$: m_3,m_4,m_2,m_1
$P(m_3)$: w_2,w_1,w_3,w_4	$P(w_3)$: m_3,m_2,m_1,m_4
$P(m_4)$: w_3,w_4,w_2,w_1	$P(w_4)$: m_1,m_3,m_2,m_4

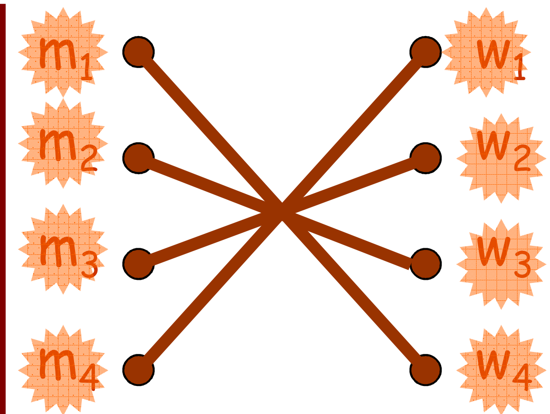


Now w_1 proposes to m_4 m_4 is free, so w_1 and w_4 get engaged

Now w_4 proposes to m_1 m_1 is free, so m_1 and w_4 get engaged.

Example 1.Gale-Shapley algorithm with women proposing

$P(m_1)$: w_1, w_2, w_3, w_4 | $P(w_1)$: m_3, m_4, m_2, m_1
$P(m_2)$: w_1, w_4, w_2, w_3 | $P(w_2)$: m_3, m_4, m_2, m_1
$P(m_3)$: w_2, w_1, w_3, w_4 | $P(w_3)$: m_3, m_2, m_1, m_4
$P(m_4)$: w_3, w_4, w_2, w_1 | $P(w_4)$: m_1, m_3, m_2, m_4



Final stable matching:

$$\mu_W = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_4 & w_3 & w_2 & w_1 \end{pmatrix}$$

Compare with the **matching obtained with men proposing**

Theorem 2. All possible executions of Gale-Shapley algorithm (with men as proposers) yield the same stable matching and in this stable matching each man has the best partner that he can have in any stable matching.

Theorem 3. In the man-optimal stable matching, each woman has the worst partner that she can have in any stable matching.

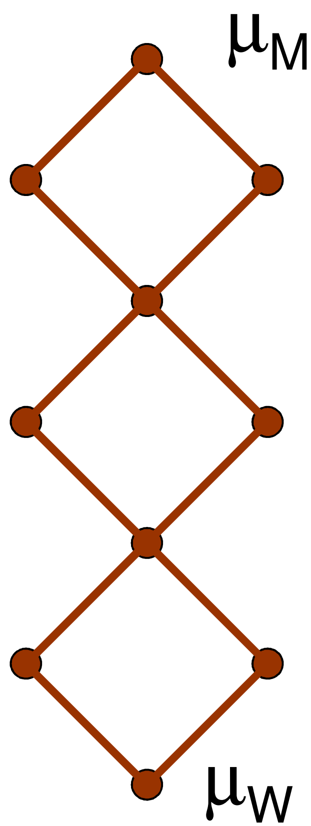
Structure of the set of stable marriages

$P(m_1)$: w_1, w_2, w_3, w_4 — $P(w_1)$: m_4, m_3, m_2, m_1

$P(m_2)$: w_2, w_1, w_4, w_3 — $P(w_2)$: m_3, m_4, m_1, m_2

$P(m_3)$: w_3, w_4, w_1, w_2 — $P(w_3)$: m_2, m_1, m_4, m_3

$P(m_4)$: w_4, w_3, w_2, w_1 — $P(w_4)$: m_1, m_2, m_3, m_4



Matching μ_M:

$$\mu_M = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix}$$

Matching μ_W:

$$\mu_W = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_4 & w_3 & w_2 & w_1 \end{pmatrix}$$

- Stable matchings form a structure called **lattice**
- an SM instance with *n* men and women may admit 2^{n-1} stable matchings
- efficient algorithms to find a stable matching fulfilling additional optimality criterion

Incomplete preference lists and different size of two sets (SMI)

$P(m_1)$: w_4, w_1	$P(w_1)$: m_4, m_1, m_2, m_3
$P(m_2)$: w_2, w_1, w_4	$P(w_2)$: m_3, m_2, m_4
$P(m_3)$: w_2, w_4, w_3	$P(w_3)$: m_1, m_3
$P(m_4)$: w_1, w_4, w_2	$P(w_4)$: m_4, m_1, m_3, m_2

- Gale-Shapley naturaly extended to this case

Theorem 4. In a the stable marriage instance with unacceptable partners, the men and women are each partitioned into two sets: those that have partners in all stable matchings and those that have partners in none.

Ties and incomplete preference lists SMTI

$P(m_1)$: w_4, w_1, w_3 $P(w_1)$: m_4, m_1, m_2, m_3
$P(m_2)$: w_2, w_1, w_4 $P(w_2)$: $(m_3, m_2), m_4$
$P(m_3)$: w_2, w_4, w_3 $P(w_3)$: m_1, m_3
$P(m_4)$: w_1, w_4, w_2 $P(w_4)$: m_4, m_1, m_3, m_2

- ties indicated by brackets
- the best choices for w_2 are m_3 and m_2
- **Algorithm:** resolve ties arbitrarily and use GS algorithm
- A matching is **super-stable** if it is stable in **every** instance of SM obtained by breaking the ties
- **Homework:** show that this instance admits no super-stable matching
- existence can be decided in polynomial time (Irving 1994)

Ties and incomplete preference lists

$P(m_1)$: w_4, w_1, w_3
$P(m_2)$: w_2, w_1, w_4
$P(m_3)$: w_2, w_4, w_3
$P(m_4)$: w_1, w_4, w_2

$P(w_1)$: m_4, m_1, m_2, m_3
$P(w_2)$: $(m_3, m_2), m_4$
$P(w_3)$: m_1, m_3
$P(w_4)$: m_4, m_1, m_3, m_2

- A matching is ***weakly stable*** if it is stable for ***some*** SM instance obtained by breaking the ties
  - a weakly stable matching always exists
- A matching μ is ***weakly stable*** if there is no blocking pair $(m,w) \notin \mu$ such that
  - *m* and *w* find each other acceptable
  - *m* is unmatched or strictly prefers *w* to $\mu(m)$
  - *w* is unmatched or strictly prefers *m* to $\mu(w)$

Weakly stable matchings may have different size

$P(m_1)$: w_4, w_1, w_3
$P(m_2)$: w_2, w_1, w_4
$P(m_3)$: w_2, w_4, w_3
$P(m_4)$: w_1, w_4, w_2

$P(w_1)$: m_4, m_1, m_2, m_3
$P(w_2)$: $(m_3, m_2), m_4$
$P(w_3)$: m_1, m_3
$P(w_4)$: m_4, m_1, m_3, m_2

weakly stable matching of size 4

$P(m_1)$: w_4, w_1, w_3
$P(m_2)$: w_2, w_1, w_4
$P(m_3)$: w_2, w_4, w_3
$P(m_4)$: w_1, w_4, w_2

$P(w_1)$: m_4, m_1, m_2, m_3
$P(w_2)$: $(m_3, m_2), m_4$
$P(w_3)$: m_1, m_3
$P(w_4)$: m_4, m_1, m_3, m_2

weakly stable matching of size 3

Maximization problem

Problem MAX-SMTI:

- Instance: Preference profile P with ties and incomplete lists.
- Task: Find a maximum cardinality weakly stable matching for P.

Theorem 5. MAX-SMTI is NP-hard

Let $\mathcal{P}$ be a maximization problem and $\mathcal{A}$ an algorithm for $\mathcal{P}$.

- Denote:
  - opt(I): optimum value of problem $\mathcal{P}$ in instance I
  - $\mathcal{A}(I)$: the value output for instance I by algorithm $\mathcal{A}$

Definition. Algorithm $\mathcal{A}$ is an α-approximation algorithm for problem $\mathcal{P}$, if for each instance I of $\mathcal{P}$: $\mathcal{A}(I) \geq \alpha.\text{opt}(I)$

Easy ½ approximation algorithm

Theorem 6. For an arbitrary instance of SMTI, the size of the largest weakly stable matching is at most twice the size of the smallest.

Proof: Let μ be a weakly stable matching of max cardinality, let μ' be weakly stable matching such that $|\mu'|<|\mu|/2$.

Then there exist men $m_1,m_2,\ldots,m_p$ matched in μ but unmatched in μ', where $p>|\mu'|$.

Women matched to those men in μ are $W'=\{w_1,w_2,\ldots,w_p\}$.

Each woman $w\in W'$ must be matched in μ', otherwise $(w,\mu(w))$ is a blocking pair for μ'.

Contradiction: μ' contains more pairs than its cardinality.

Z. Király's approximation algorithm (2008-2012):

2/3 if men have strict preferences

3/5 in general case

Hospitals/Residents problem (HR)

- residents **R={r_1, r_2, …, r_n}**, hospitals **R={ h_1, h_2, …, h_m}**
- hospital h_i has ***capacity*** $\mathbf{q}_i$
- each resident ranks a subset of H in strict order of preference
- each hospital ranks its applicants in strict order of preference
- r finds h ***acceptable*** if h is on r's preference list and conversely

An allocation μ of residents to hospitals is a ***matching*** if:

- $(r,h) \in \mu \Rightarrow r,h$ find each other acceptable
- No resident has more than one post and no hospital exceeds its capacity

Matching μ is ***stable*** if μ admits no ***blocking pair*** **(r,h):**

- r, h find each other acceptable and
- either r is unmatched in μ or r prefers h to his/her allocated hospital in μ and
- either h is undersubscribed in μ or h prefers r to its worst resident assigned in μ

Unstable matching

$P(r_1)$: h_2 h_1
$P(r_2)$: h_1 h_2
$P(r_3)$: h_1 h_3
$P(r_4)$: h_2 h_3
$P(r_5)$: h_2 h_1
$P(r_6)$: h_1 h_2

Resident preferences

Each hospital has 2 posts

$P(h_1)$: r_1 r_3 r_2 r_5 r_6
$P(h_2)$: r_2 r_6 r_1 r_4 r_5
$P(h_3)$: r_4 r_3

Hospital preferences

This matching is unstable as (r_2,h_1) is a blocking pair.

Homework: find 2 other blocking pairs

Algorithms for HR

- Hospital-oriented Gale-Shapley algorithm
- Resident-oriented Gale-Shapley algorithm

The Rural Hospitals Theorem. For any instance of HR:

1. each hospital is assigned the same number of residents in all stable matchings
2. the same residents are assigned in all stable matchings.
3. any hospital that is undersubscribed in one stable matching is assigned exactly the same residents in all stable matchings.

Hospital-oriented Gale-Shapley algorithm

$P(r_1)$: h_2 h_1
$P(r_2)$: h_1 h_2
$P(r_3)$: h_1 h_3
$P(r_4)$: h_2 h_3
$P(r_5)$: h_2 h_1
$P(r_6)$: h_1 h_2

Resident preferences

Each hospital has 2 posts

$P(h_1)$: r_1 r_3 r_2 r_5 r_6
$P(h_2)$: r_2 r_6 r_1 r_4 r_5
$P(h_3)$: r_4 r_3

Hospital preferences

Hospital h_1 proposes to resident r_1.
Hospital h_1 proposes to resident r_3. Pair (h_3,r_3) deleted.

Hospital-oriented Gale-Shapley algorithm

$P(r_1)$: h_2 h_1
$P(r_2)$: h_1 h_2
$P(r_3)$: h_1 h_3
$P(r_4)$: h_2 h_3
$P(r_5)$: h_2 h_1
$P(r_6)$: h_1 h_2

Resident preferences

Each hospital has 2 posts

$P(h_1)$: r_1 r_3 r_2 r_5 r_6
$P(h_2)$: r_2 r_6 r_1 r_4 r_5
$P(h_3)$: r_4 r_3

Hospital preferences

Hospital h_2 proposes to resident r_2.
Hospital h_2 proposes to resident r_6.
Hospital h_3 proposes to resident r_4.

Example: residents oriented algorithm

$P(r_1)$: h_2 h_1
$P(r_2)$: h_1 h_2
$P(r_3)$: h_1 h_3
$P(r_4)$: h_2 h_3
$P(r_5)$: h_2 h_1
$P(r_6)$: h_1 h_2

Resident preferences

Each hospital has 2 posts

$P(h_1)$: r_1 r_3 r_2 r_5 r_6
$P(h_2)$: r_2 r_6 r_1 r_4 r_5
$P(h_3)$: r_4 r_3

Hospital preferences

Resident r_1 proposes to hospital h_2.
Resident r_2 proposes to hospital h_1.

Example: residents oriented algorithm

$P(r_1)$: h_2 h_1
$P(r_2)$: h_1 h_2
$P(r_3)$: h_1 h_3
$P(r_4)$: h_2 h_3
$P(r_5)$: h_2 h_1
$P(r_6)$: h_1 h_2

Resident preferences

Each hospital has 2 posts

$P(h_1)$: r_1 r_3 r_2 r_5 r_6
$P(h_2)$: r_2 r_6 r_1 r_4 r_5
$P(h_3)$: r_4 r_3

Hospital preferences

Resident r_3 proposes to hospital h_1. Hospital h_1 deletes r_5 and r_6.
Resident r_4 proposes to hospital h_2. Hospital h_2 deletes r_5
Resident r_6 proposes to hospital h_2. Hospital h_2 deletes r_4.
Resident r_4 proposes to hospital h_3.

Hospitals/Residents problem with couples

- residents $r_1, r_2, \ldots, r_n$, hospitals $h_1, h_2, \ldots, h_m$
- hospital *h* has ***capacity*** **q(h)**
- hospital ranks its applicants in strict order of preference
- residents: are single and in couples
  - no resident may be a member of more than one couple
  - a single resident ranks a subset of hospitals in strict order of preference
  - each couple (r,s) provides a joint preference list, each entry is an ordered pair (h,k) of (not necessarily distinct) hospitals

Stability

A matching μ is **unstable** if it is blocked in either of the three ways:

- by a hospital h and a single resident r
- by a hospital h and a resident r of a couple with s:
  - r is acceptable to h
  - (r,s) prefers (h, μ(s)) to (μ(r), μ(s))
  - h is either undersubscribed or prefers r to at least one of its asigned residents in μ
- by a couple (r,s) and hospitals (not necessarily distinct) $h_1 \neq \mu(r)$ and $h_2 \neq \mu(s)$.

Hospitals/Residents problem with couples - example 1

Hospitals' preferences:

P(h): r, t, s, u

P(k): s, r, t, u

Residents' preferences

P(r,s): (h,h), (k,k), (k,h), (h,k)

P(t): h, k

P(u): k, h

matching:
$\mu(h) = \{r,s\}$,
$\mu(k) = \{t,u\}$

Blocked by:
hospital and single resident:
h and t

*capacities of both hospitals are 2

Hospitals/Residents problem with couples - example 2

Hospitals' preferences:

P(h): r, t, s, u

P(k): s, r, t, u

Residents' preferences

P(r,s):(h,h), (k,k), (k,h), (h,k)

P(t): h, k

P(u): k, h

matching:

μ(h)= {r,t},

μ(k)= {s,u}

Blocked by:

hospital and married resident:

k and r

*capacities of both hospitals are 2

Hospitals/Residents problem with couples - example 3

Hospitals' preferences:

P(h): r, t, s, u

P(k): s, r, t, u

Residents' preferences

P(r,s): (h,h), (k,k), (k,h), (h,k)

P(t): h, k

P(u): k, h

matching:
μ(h)= {t,u},
μ(k)= {r,s}

Blocked by:
two hospitals and a couple:
(h,h) and (r,s)

Homework: show there is no stable matching in this example

HR with couples - computational complexity

Theorem (Ronn 1990). The problem of deciding whether an instance of the hospitals/residents matching problem with couples admits a stable matching is NP-complete, even if there are no single residents and each hospital has capacity 1.

Loss of structure (Aldershof and Carducci, 1996)

- *Even if an instance of the couples problem has a stable matching, it may not have a hospital optimal or student optimal stable matching.*
- *There may be stable matchings which leave diflerent numbers of positions unfilled.*

Approximation algorithms, heuristics and empirical studies:

- Marx and Schlotter (2011): parameterized complexity and local search
- Biró, Manlove, McBride (2014): Integer programming

Matching medical students to pairs of hospitals: specific for UK market

An instance of the HR problem with pairs of hospitals (HR2H):

- a list of students $R=\{r_1,r_2,\ldots,r_n\}$ and for each one
  - preference list of medical units (if he seeks such a place)
  - preference list of surgical units (if he seeks such a place)
  - optional seasonal preference
- a list of medical units $M=\{m_1,m_2,\ldots,m_p\}$ and surgical units $S=\{s_1,s_2,\ldots,s_q\}$ and:
  - for each m_i the number of posts offered in each half year are x_i^1,x_i^2
  - for each s_j the number of posts offered in each half year are y_j^1,y_j^2
  - for each unit: a single preference list of students to which it wishes to offer a position

The algorithm for HR2H

1. **stable matchings are found:**
   - of medical candidates to medical units based on the total number of posts in both half-years
   - of surgical candidates to surgical units based on the total number of posts in both half-years
2. **with the input of the previous step, an allocation of each matched (student,unit) pair to a half-year (called valid) so that**
   - each student who is matched to two posts has them scheduled in different half-years
   - for each unit and for each half-year, the number of allocated students does not exceed the number of posts for that half-year
   - the number of satisfied seasonal preferences is as large as possible

Flow algorithm for a valid assignment (Irving 1998)

Practical placement of teachers

Traditionally, upper elementary and lower secondary teachers in Slovakia

- specialize in two subjects (MF, IB, SjG,...)
- practical placements at schools during study

- ideally at different types of schools
- each student needs an approved supervising teacher
- university/faculty provides a list of teaching schools
- often the desire to have all schools in the site of university/faculty
- but schools in other towns of the region are used too
- some schools are unacceptable for a student (e.g. commuting)
- students practice **both subjects simultaneously at the same school**

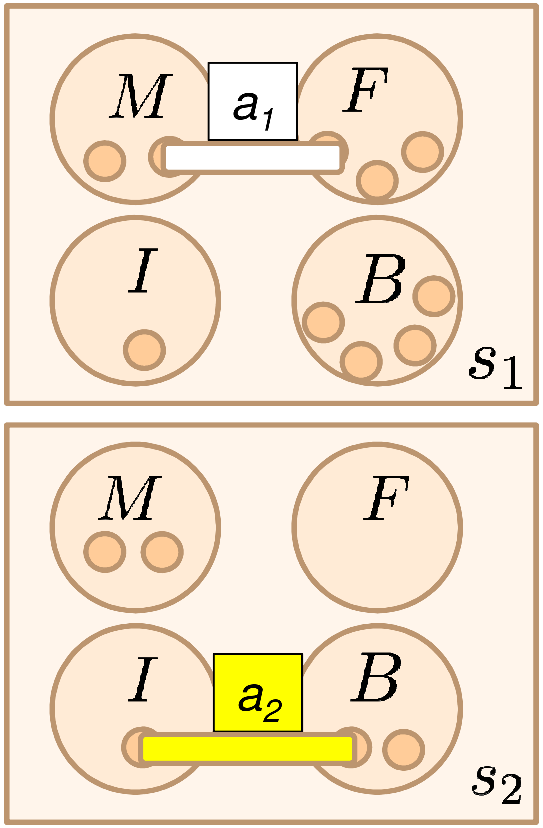
Formal model: TAP

- set $P = \{M, F, B, I, \dots\}$ of subjects
- set S of schools

	c_M	c_F	c_B	c_I	$\dots$
s_1	2	3	4	1	$\dots$
s_2	1	0	2	1	$\dots$
$\vdots$					

- set A of applicants = teachers

	$\mathbf{p}(a_i)$	$\mathbf{s}(a_i)$
a_1	MF	$\{s_1, s_2\}$
a_2	IB	$\{s_1, s_2, s_5, \dots\}$
$\vdots$	$\vdots$	$\vdots$



Computational complexity

An assignment of students to schools is *feasible* if:

- each student is assigned to an acceptable school
- for each school s and each subject p: the number of students assigned to s whose specialization includes p does not exceed the capacity $c_p(s)$ of school s in subject p.

Theorem. In the TAP problem with inseparable subjects the problem of deciding whether a full assignment exists is NP-complete even in the cases when

- there are 3 subjects and no partial capacity exceeds 2;
- there are 4 subjects and no partial capacity exceeds 1.

Integer linear program for TAP

Students $A = \{a_1, \ldots, a_n\}$, schools $S = \{s_1, \ldots, s_m\}$, subjects $P = \{p_1, \ldots, p_k\}$
Student a_i has a k-vector $\mathbf{y}$: $y_{ir} = 1$ iff a_i studies subject p_r
Student a_i has an ordered list of acceptable schools of length $\ell(a_i)$
Let $s(a_i, \rho)$ be the school in the ρ-th place of a_i's list
Binary variables $x_{i,\rho}$ for each a_i and each $\rho = 1, 2, \ldots, \ell(a_i) + 1$

Interpretation: $x_{i,\rho} = \begin{cases} 1 & \text{if } a_i \text{ is assigned to the schools in position } \rho \\ 0 & \text{otherwise} \end{cases}$

Cost function:

$$\sum_{i=1}^{n} \sum_{\rho=1}^{\ell(a_i)} x_{i\rho} \rightarrow max$$

Constraints:

$$\sum_{i=1}^{\ell(a_i)+1} x_{i\rho} = 1$$

$$\sum_{i=1}^{n} \sum_{\rho=1}^{\ell(a_i)} \{x_{i\rho} : s(a_i, \rho) = s_j \ \& \ y_{i\rho} = 1\} \leq c_\rho(s_j)$$

$$x_{i\rho} \in \{0, 1\}$$

UPJŠ in numbers 2014-2015

	Science faculty						Philosophical faculty							
	M	F	B	C	I	G	Sj	Aj	Nj	Lj	O	Ps	E	D
# students	13	9	43	21	4	35	31	14	22	1	21	22	12	28

Usual practice: students of Ps are allocated to E or Ov.

year	# of students	# of schools	# of assigned	time
2015	82	59	82	8 sec
2014	138	197	137	21 sec
2014	138	59	120	6 minutes
2014+2015	220	197	208	13 minutes
2014+2015	220	59	no result	$>$ 7 hours